Honors Discrete Mathematics

Instructor: Alexander Razborov, University of Chicago
razborov@math.uchicago.edu
Course Homepage: www.cs.uchicago.edu/~razborov/teaching/autumn20.html

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   Prove all of your answers with reasonable degree of mathematical
rigor.  If you work with others put their names clearly at the top
of the assignment.  Everyone must turn in their own independently
written solutions.  Shopping for solutions on the Internet is strongly
discouraged:  if you do it nonetheless, you *must* cite your source
and, as the very least, explain the solution in your own words.
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Homework 1, due October 14

1. Describe explicitly those positive integers n for which

$$1! + 2! + \ldots + n! \leq \frac{1}{5}(n+1)!.$$

2. Let $n \geq 1$ and consider the "triangular" set of positive integer points

$$\{(x, y) \in \mathbb{N}^2 \mid x + y \leq n\}$$

on the plane. What is the minimum number of lines needed to cover all these points?

3. (a) Prove that every ordering $\preceq$ of a finite set S that is not total has at least one partial extension $(S, \preceq^*)$ such that $|\preceq^*| = |\preceq| + 1$.

   (b) Give an example of an infinite partially ordered set $(S, \preceq)$ that is not total and such that for every non-trivial partial extension $(S, \preceq^*)$, the number of pairs (s, t) such that $s \preceq^* t$ but $s \not\preceq t$ is infinite.

4. Let $\mathbb{PSF}$ be the set of all positive square-free integers, i.e. those that are not divisible by any integer of the form m^2 with $m > 1$. Prove that the induced partial order $(\mathbb{PSF},\ |)$ is isomorphic to the partial order $(\mathcal{P}_{<\omega}(\mathbb{N}),\ \subseteq)$, where $P_{<\omega}(\mathbb{N})$ consists of all *finite* subsets of positive integers.

   `Note. You can assume both parts of the Fundamental Theorem of Arithmetic (existence and uniqueness).`

5. Construct an infinite linearly ordered set S and a finite non-empty linearly ordered set T such that $S + T$ is isomorphic to $T + S$.

   `For extra credit:` May this happen if S is well-ordered?