Honors Discrete Mathematics

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   Prove all of your answers with reasonable degree of mathematical
rigor.  If you work with others put their names clearly at the top
of the assignment.  Everyone must turn in their own independently
written solutions.  Shopping for solutions on the Internet is strongly
discouraged:  if you do it nonetheless, you *must* cite your source
and, as the very least, explain the solution in your own words.
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Homework 2, due October 21

1. Let α be an ordinal number. Denote by X the set of those positive integers n for which α can be represented in the form $\alpha = \beta + n$. Prove that X has a *maximal* element provided it is non-empty.

2. The *least common multiple* $\mathrm{lcm}(a, b)$ is defined as the smallest positive integer that is divisible by both a and b.

   Prove the inequality

$$\mathrm{lcm}(ab, c) \geq \frac{\mathrm{lcm}(a, c) \cdot \mathrm{lcm}(b, c)}{c}; \; a, b, c \in \mathbb{N}.$$

3. Prove that $\gcd(n^2 - n + 1, n^2 + n - 1) = 1$ for any $n \in \mathbb{Z}$.

4. Consider the structure $(\mathbb{Z}, \oplus, \otimes)$ in which the "addition" $\oplus$ and the "multiplication" $\otimes$ are defined as follows:

$$\begin{aligned} x \oplus y &\stackrel{\text{def}}{=} \min(x, y) \\ x \otimes y &\stackrel{\text{def}}{=} x + y. \end{aligned}$$

   What axioms of a commutative ring does this structure satisfy?

5. For two ideals I and J in a commutative ring, let

$$I + J \stackrel{\text{def}}{=} \{a + b \,|\, a \in I,\ b \in J\}$$
$$I \cdot J \stackrel{\text{def}}{=} \{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \,|\, n \in \mathbb{N},\ a_1, \ldots, a_n \in I,\ b_1, \ldots, b_n \in J\}\,.$$

Prove that $I{+}J$ and $I{\cdot}J$ are also ideals and that $I{\cdot}(J{+}K) = I{\cdot}J{+}I{\cdot}K$ for any three ideals I, J, K.