Honors Discrete Mathematics

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Prove all of your answers with reasonable degree of mathematical rigor. If you work with others put their names clearly at the top of the assignment. Everyone must turn in their own independently written solutions. Shopping for solutions on the Internet is strongly discouraged: if you do it nonetheless, you *must* cite your source and, as the very least, explain the solution in your own words.

Homework 3, due October 28

1. Prove that for any positive integer \( n \) ending (in its decimal representation) with 7, the ideal generated by \( n \) and \( 1 + \sqrt{11} \) in \( \mathbb{Z}[\sqrt{11}] \) is trivial (coincides with the whole ring).

2. Construct two equivalence relations \( \approx_1 \) and \( \approx_2 \) on the set \([2020]\) such that:

   (a) Every equivalence class of either \( \approx_1 \) or \( \approx_2 \) has exactly two elements;

   (b) Every equivalence relation \( \approx \) that is coarser\(^1\) than both \( \approx_1 \) and \( \approx_2 \) is trivial, i.e. consists of all possible 2020\(^2\) pairs.

\(^1\)This means that any equivalence class of either \( \approx_1 \) or \( \approx_2 \) is contained in an equivalence class of \( \approx \).

\(^2\)Counts the number of ways two elements can be paired.
3. Solve the following system of congruences:

\[
\begin{align*}
    x &\equiv 1 \pmod{2} \\
    x &\equiv 2 \pmod{3} \\
    x &\equiv 1 \pmod{5} \\
    x &\equiv 1 \pmod{7} \\
    x &\equiv 5 \pmod{9}.
\end{align*}
\]

4. The product\(^2\) of partially ordered sets \((S_1, \preceq_1), \ldots, (S_n, \preceq_n)\) is the partial order \(\preceq\) on the Cartesian product \(S_1 \times S_2 \times \cdots \times S_n\) given by \((s_1, \ldots, s_n) \preceq (t_1, \ldots, t_n)\) iff \(s_i \preceq_i t_i\) for all \(i \in [n]\).

Let \((S, \preceq)\) be a linearly ordered set with \(m\) elements and \((S^n, \preceq^n)\) be its \(n\)th power in the above sense. Give a closed form expression for \(|\preceq^n|\).

5. Let us call a function \(f : X \to Y\) good if for any function \(g : Z \to Y\) with \(|Z| \leq 13\) there exists an injective function \(h : Z \to X\) such that \(g\) decomposes as \(g = f \circ h\).

Describe the set of those positive integers \(n\) for which there exists a good function \(f : [1000] \to [n]\).