Honors Discrete Mathematics

Instructor: Alexander Razborov, University of Chicago
razborov@math.uchicago.edu
Course Homepage: www.cs.uchicago.edu/~razborov/teaching/autumn20.html

Autumn Quarter, 2020

```
   Prove all of your answers with reasonable degree of mathematical
rigor.  If you work with others put their names clearly at the top
of the assignment.  Everyone must turn in their own independently
written solutions.  Shopping for solutions on the Internet is strongly
discouraged:  if you do it nonetheless, you *must* cite your source
and, as the very least, explain the solution in your own words.
```

Homework 3, due October 28

1. Prove that for any positive integer n ending (in its decimal representation) with 7, the ideal generated by n and $1+\sqrt{11}$ in $\mathbb{Z}[\sqrt{11}]$ is trivial (coincides with the whole ring).

2. Construct two equivalence relations $\approx_1$ and $\approx_2$ on the set $[2020]$ such that:

   (a) Every equivalence class of either $\approx_1$ or $\approx_2$ has exactly two elements;

   (b) Every equivalence relation $\approx$ that is *coarser*[1] than both $\approx_1$ and $\approx_2$ is trivial, i.e. consists of all possible 2020^2 pairs.

[1]This means that any equivalence class of either $\approx_1$ or $\approx_2$ is contained in an equivalence class of $\approx$.

3. Solve the following system of congruences:

$$\begin{cases} x \equiv 1 \pmod{2} \\ x \equiv 2 \pmod{3} \\ x \equiv 1 \pmod{5} \\ x \equiv 1 \pmod{7} \\ x \equiv 5 \pmod{9}. \end{cases}$$

4. The *product*[2] of partially ordered sets $(S_1, \preceq_1), \ldots, (S_n, \preceq_n)$ is the partial order $\preceq$ on the Cartesian product $S_1 \times S_2 \ldots \times S_n$ given by $(s_1, \ldots, s_n) \preceq (t_1, \ldots, t_n)$ iff $s_i \preceq_i t_i$ for all $i \in [n]$.

   Let $(S, \leq)$ be a linearly ordered set with m elements and $(S^n, \leq^n)$ be its nth power in the above sense. Give a closed from expression for $| \leq^n |$.

5. Let us call a function $f : X \longrightarrow Y$ *good* if for any function $g : Z \longrightarrow Y$ with $|Z| \leq 13$ there exists an injective function $h : Z \rightarrowtail X$ such that g decomposes as $g = f \circ h$.

   Describe the set of those positive integers n for which there exists a good function $f : [1000] \longrightarrow [n]$.

[2] Not to be confused with the lexicographic product we did in class.