Honors Discrete Mathematics

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Prove all of your answers with reasonable degree of mathematical
rigor. If you work with others put their names clearly at the top
of the assignment. Everyone must turn in their own independently
written solutions. Shopping for solutions on the Internet is strongly
discouraged: if you do it nonetheless, you *must* cite your source
and, as the very least, explain the solution in your own words.
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Homework 4, due November 4

1. Let us call a function $f : X \longrightarrow Y$ *co-good* if for any function $g : X \longrightarrow Z$ with $|Z| \leq 13$ there exists a surjective function $h : Y \longrightarrow Z$ such that g decomposes as $g = h \circ f$.

   Describe the set of those positive integers n for which there exists a co-good function $f : [1000] \longrightarrow [n]$.

2. The ℓ_∞-*norm* of a vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is defined as $\ell_\infty(\mathbf{x}) \stackrel{\text{def}}{=} \max_{1 \leq i \leq n} |x_i|$.

   How many ordered triples $\langle \mathbf{x}, \mathbf{y}, \mathbf{z} \rangle \in (\{-1, 1\}^n)^3$ of (± 1) n-dimensional vectors are there with the property $\ell_\infty(\mathbf{x} + \mathbf{y} + \mathbf{z}) \leq 1$?

3. Prove that
$$\sum_{k=0}^{n} \sum_{\ell=0}^{n} \binom{n}{k} \binom{n}{\ell} \binom{n}{k+\ell} = \binom{3n}{n}.$$

4. Let A_x ($x \in \mathbb{Z}_7^*$) be finite sets of cardinality 100 each such that for any $x \in \mathbb{Z}_7^*$, A_x is disjoint with A_{-x} and A_{3x}, $|A_x \cap A_{2x}| = 10$ and $|A_x \cap A_{2x} \cap A_{4x}| = 1$. What is the cardinality of their union?

5. Give a closed form expression for the number $M(m)$ of equivalence relations on $[2m]$ in which every class has exactly 2 elements.