Honors Discrete Mathematics

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Prove all of your answers with reasonable degree of mathematical
rigor. If you work with others put their names clearly at the top
of the assignment. Everyone must turn in their own independently
written solutions. Shopping for solutions on the Internet is strongly
discouraged: if you do it nonetheless, you *must* cite your source
and, as the very least, explain the solution in your own words.
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Homework 5, due November 11

1. Prove the inequality
$$S(m, n-2) \le \binom{n}{3} S(m, n) \ (3 \le n,\ m \ge 2n-3).$$

2. How many ways are there to choose seven cards out of a standard deck of 52 cards in such a way that all face cards (J,Q,K) among the chosen are red (Diamonds or Hearts)? Order does not matter.

   `Note. To qualify for full credit, the final answer in this and the next problem must be an integer.`

3. How many solutions are there to the inequality
$$x_1 + x_2 + \ldots + x_7 \le 86,$$
where x_i are **odd** positive integers? Order matters.

4. Let $m \ge k \ge 1$ be fixed integers. Call a partition of m (that is, a distribution of m indistinguishable balls into an unspecified number

of indistinguishable boxes) *fair* if every non-empty box has at least k balls in it. Call it *balanced* if the k most populous boxes contain the same number of balls.

Prove that the number of fair partitions is equal to the number of balanced ones.

5. $n^2 + 1$ points are chosen inside the unit square.

   (a) Prove that there is a pair of them at the (Euclidean) distance $\le \frac{\sqrt{2}}{n}$ of each other.

   (b) Prove that this bound is tight for $n = 2$, i.e. that there are five points with all pairwise distances $\ge \frac{\sqrt{2}}{2}$.

   `For extra credit:` Is it tight (in the same sense as in item b) above) for $n = 3$?