Honors Discrete Mathematics

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Prove all of your answers with reasonable degree of mathematical
rigor.  If you work with others put their names clearly at the top
of the assignment.  Everyone must turn in their own independently
written solutions.  Shopping for solutions on the Internet is strongly
discouraged:  if you do it nonetheless, you *must* cite your source
and, as the very least, explain the solution in your own words.
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Homework 7, due December 2

1. Let $n \geq 2$. For a permutation $\sigma : [n] \longrightarrow [n]$, define its *number of descents* $d(\sigma)$ as $|\{i \in [n-1] \mid \sigma(i+1) < \sigma(i)\}|$.

   Compute, as a closed form expression, the variance of $d(\sigma)$, where σ is picked uniformly from the set of all permutations S_n.

2. Assume that $d_1, \ldots, d_n$ is the degree sequence of a simple graph. Prove that the following sequences are also degree sequences of a simple graph (order does not matter):

   (a) $n-1-d_1, n-1-d_2, \ldots, n-1-d_n$;

   (b) $2d_1, 2d_1, 2d_2, 2d_2, \ldots, 2d_n, 2d_n$;

   (c) $d_1+1, d_1+1, d_2+1, d_2+1, \ldots, d_{k-1}+1, d_{k-1}+1, d_k, d_k, \ldots, d_n, d_n$.

3. Let $a \geq 2$ be an integer and $b \in \mathbb{Z}_a$, $b \neq 0$. Consider the graph $G = (V, E)$, where $V = \mathbb{Z}_a$ and E consists of all edges of the form $(x, x+b)$, $x \in \mathbb{Z}_a$. How many connected components does G have?

4. The *cartesian product* of two graphs $G = (V, E)$ and $H = (W, F)$ is defined as the graph $G \Box H$ with the vertex set $V \times W$ and the set of edges

$$\{ (\langle v_1, w\rangle, \langle v_2, w\rangle) \mid (v_1, v_2) \in E,\ w \in W \} \cup \{ (\langle v, w_1\rangle, \langle v, w_2\rangle) \mid v \in V,\ (w_1, w_2) \in F \}.$$

The *tensor product* $G \times H$ has the same vertex set but the set of edges is

$$\{ (\langle v_1, w_1\rangle, \langle v_2, w_2\rangle) \mid (v_1, v_2) \in E,\ (w_1, w_2) \in F \}.$$

Finally, the *lexicographic product* $G[H]$ still has the vertex set $V \times W$ and the set of edges is

$$\{ (\langle v_1, w_1\rangle, \langle v_2, w_2\rangle) \mid (v_1, v_2) \in E;\ w_1, w_2 \in W \} \cup \{ (\langle v, w_1\rangle, \langle v, w_2\rangle) \mid v \in V,\ (w_1, w_2) \in F \}.$$

For which of these products the following statement

if G and H are bi-partite then so is their product

is true?

5. Prove that for any graph G on n vertices,

$$\alpha(G)\omega(G) \leq \begin{cases} \frac{(n+1)^2}{4}, & n \text{ odd} \\ \frac{n(n+2)}{4}, & n \text{ even.} \end{cases}$$