Homework 2, due October 20

1. Describe explicitly all those linearly ordered sets $S$ for which:

   (a) The lexicographic product $S \times_{\text{lex}} \mathbb{Z}$ is isomorphic to $\mathbb{Z}$;
   (b) The lexicographic product $\mathbb{Z} \times_{\text{lex}} S$ is isomorphic to $\mathbb{Z}$.

2. The least common multiple $\text{lcm}(a, b)$ is defined as the smallest positive integer that is divisible by both $a$ and $b$.

   Prove the identity

   $$\text{lcm}(\gcd(a, b), c) = \gcd(\text{lcm}(a, c), \text{lcm}(b, c)).$$

3. Consider the equation

   $$\frac{1}{a} + \frac{1}{b} = \frac{5}{2021}.$$

   (a) Solve it in positive integers $a, b$.
   (b) Prove that there are no other solutions (up to a permutation of its components).
4. Prove that for any positive integer $n$, $\gcd(n^2 + n + 1, 3n + 1) \in \{1, 7\}$.

5. For two ideals $I$ and $J$ in a commutative ring $R$, let

\[
I + J \overset{\text{def}}{=} \{ a + b \mid a \in I, \ b \in J \}
\]
\[
I \cdot J \overset{\text{def}}{=} \{ a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \mid n \in \mathbb{N}, \ a_1, \ldots, a_n \in I, \ b_1, \ldots, b_n \in J \}.
\]

Prove that $I + J$ and $I \cdot J$ are also ideals and that $I \cdot (J + K) = I \cdot J + I \cdot K$ for any three ideals $I, J, K$. 