Honors Discrete Mathematics

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  Prove all of your answers with reasonable degree of mathematical
rigor (feel free to ask us when in doubt).  If you work with others
put their names clearly at the top of the assignment, everyone must
turn in their own independently written solutions.  Shopping for
solutions on the Internet is strongly discouraged.  But if you accidentally
stumble across it, then it will be accepted as long as you cite
the source and explain the solution in your own words.
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Homework 1, due October 12

1. Let $x_1 = x_2 = 1$, $x_3 = 4$, and $x_{n+3} = 2x_{n+2} + 2x_{n+1} - x_n$ for $n \geq 1$. Prove that x_n is a square for all $n \geq 1$.

2. Let $n \geq 1$ and consider the "triangular" set of positive integer points

$$\left\{ (x, y) \in \mathbb{N}^2 \,|\, x + y \leq n \right\}$$

   on the plane. What is the minimum number of lines needed to cover all these points?

3. A function $f : \mathbb{N} \longrightarrow \mathbb{N}$ satisfies the following two properties:

$$\begin{aligned} f(n) &= n - 7 \quad \text{if} \quad n > 49 \\ f(n) &= f(f(n+8)) \quad \text{if} \quad n \leq 49. \end{aligned}$$

   Compute $f(11)$.

4. Consider the (infinite) set S of all real (that is, $a_n \in \mathbb{R}$) sequences $a = (a_1, \ldots, a_n, \ldots)$. Define a binary relation $\preceq$ on S by letting $a \preceq b$ if and only if there exist absolute constants $C_1 > 0,\ C_2 > 0$ such that $\forall n \in \mathbb{N}(a_n \leq C_1 b_n + C_2)$.

   (a) What axioms of total order does this relation satisfy? Give a proof when a property is satisfied and a counterexample when it is not.

   (b) Will the answer change if we drop the positivity restriction $C_1 > 0$ thus allowing C_1 to be negative? Will it change if we drop the condition on C_2 instead (still requiring $C_1 > 0$)?

5. Prove that for every $n \geq m \geq 1$ there exists a partial ordering of $[n]$ that has *exactly* m different total extensions.