Honors Discrete Mathematics

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Prove all of your answers with reasonable degree of mathematical rigor (feel free to ask us when in doubt). If you work with others put their names clearly at the top of the assignment, everyone must turn in their own independently written solutions. Shopping for solutions on the Internet is strongly discouraged. But if you accidentally stumble across it, then it will be accepted as long as you cite the source and explain the solution in your own words.

Homework 1, due October 12

1. Let $x_1 = x_2 = 1$, $x_3 = 4$, and $x_{n+3} = 2x_{n+2} + 2x_{n+1} - x_n$ for $n \geq 1$.
   Prove that $x_n$ is a square for all $n \geq 1$.

2. Let $n \geq 1$ and consider the “triangular” set of positive integer points
   \[
   \{(x, y) \in \mathbb{N}^2 \mid x + y \leq n \}
   \]
   on the plane. What is the minimum number of lines needed to cover all these points?

3. A function $f : \mathbb{N} \rightarrow \mathbb{N}$ satisfies the following two properties:
   \[
   f(n) = n - 7 \quad \text{if} \quad n > 49
   \]
   \[
   f(n) = f(f(n + 8)) \quad \text{if} \quad n \leq 49.
   \]
   Compute $f(11)$. 

4. Consider the (infinite) set $S$ of all real (that is, $a_n \in \mathbb{R}$) sequences $a = (a_1, \ldots, a_n, \ldots)$. Define a binary relation $\preceq$ on $S$ by letting $a \preceq b$ if and only if there exist absolute constants $C_1 > 0$, $C_2 > 0$ such that $\forall n \in \mathbb{N}(a_n \leq C_1 b_n + C_2)$.

(a) What axioms of total order does this relation satisfy? Give a proof when a property is satisfied and a counterexample when it is not.

(b) Will the answer change if we drop the positivity restriction $C_1 > 0$ thus allowing $C_1$ to be negative? Will it change if we drop the condition on $C_2$ instead (still requiring $C_1 > 0$)?

5. Prove that for every $n \geq m \geq 1$ there exists a partial ordering of $[n]$ that has exactly $m$ different total extensions.