Honors Discrete Mathematics

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Autumn Quarter, 2022

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   Prove all of your answers with reasonable degree of mathematical
rigor (feel free to ask us when in doubt).  If you work with others
put their names clearly at the top of the assignment, everyone must
turn in their own independently written solutions.  Shopping for
solutions on the Internet is strongly discouraged.  But if you accidentally
stumble across it, then it will be accepted as long as you cite
the source and explain the solution in your own words.
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Homework 3, due October 26

1. Let A be an arbitrary infinite set of integers such that any $x \in A$ divides a sufficiently large power of 10. Prove that there exist two distinct $x, y \in A$ such that $x|y$.

2. Let X be a non-empty set and Λ be the collection of all its subsets. Let us try to define the structure of a commutative ring on Λ by letting $0 \stackrel{\text{def}}{=} \emptyset,\ 1 \stackrel{\text{def}}{=} X,\ A+B \stackrel{\text{def}}{=} A \cup B,\ A \cdot B \stackrel{\text{def}}{=} A \cap B$.

   (a) Which axioms of a commutative ring will fail for this structure?

   (b) Find one natural axiom that will hold in Λ but will not hold in any commutative ring.

3. Consider the following binary relation $R(a, b)$ on $\mathbb{Z}$: $R(a, b)$ holds if and only if $a^3 - b^3 = a - b$. Prove that R is an equivalence relation in which every equivalence class is finite and contains an odd number of elements.

4. Prove that the system of congruences

$$\begin{cases} x \equiv a_1 \bmod m_1 \\ x \equiv a_2 \bmod m_2 \end{cases}$$

has a solution $x \in \mathbb{Z}$ if and only if $a_1 \equiv a_2 \bmod \gcd(m_1, m_2)$.

5. Let us call a function $f : X \longrightarrow Y$ *gorgeous* if for any function $g : Z \longrightarrow Y$ with $|Z| \leq 17$ there exists an injective function $h : Z \rightarrowtail X$ such that $g = f \circ h$. Let us call f *co-gorgeous* if for any $g : X \longrightarrow Z$, again with $|Z| \leq 17$, there exists a surjective $h : Y \twoheadrightarrow Z$ such that $g = h \circ f$.

   (a) Describe the set of those integers n for which there exists a gorgeous function $f : [100] \longrightarrow [n]$.

   (b) Describe the set of those integers n for which there exists a co-gorgeous function $f : [100] \longrightarrow [n]$.