Honors Discrete Mathematics

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   Prove all of your answers with reasonable degree of mathematical
rigor (feel free to ask us when in doubt).  If you work with others
put their names clearly at the top of the assignment, everyone must
turn in their own independently written solutions.  Shopping for
solutions on the Internet is strongly discouraged.  But if you accidentally
stumble across it, then it will be accepted as long as you cite
the source and explain the solution in your own words.
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Homework 4, due November 4

1. Give a closed-form expression for the number of ordered triples $\langle A, B, C\rangle$ of subsets of $[n]$ such that $C \neq \emptyset$ and $A \cap B \subseteq C$.

2. (a) Show that for $k \leq n$ we have
$$\sum_{i=0}^{k} (-1)^i \binom{n}{i} \geq 0$$
if k is even and
$$\sum_{i=0}^{k} (-1)^i \binom{n}{i} \leq 0$$
if k is odd.

   (b) Prove the fact mentioned in class: partial sums in the inclusion-exclusion expression oscillate around the actual value.

3. How many way are there to choose five cards out of a standard deck of 52 cards in such a way that at least one suit is missing in the selection?

   `Note. To earn full credit, the final answer must be a plain integer. But once you write down a closed-form expression, you need not show intermediate results of the remaining calculations.`

4. Prove that for any integer $n \geq 1$, $(n!)^{n+1} | (n^2)!$.

5. Give a closed-form expression for the number of *integer* (that is, not necessarily positive) solutions to the equation

$$x_1 x_2 \cdots x_n = 2^a 3^b 5^c.$$

   Order matters.