Honors Discrete Mathematics

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Prove all of your answers with reasonable degree of mathematical rigor (feel free to ask us when in doubt). If you work with others put their names clearly at the top of the assignment, everyone must turn in their own independently written solutions. Shopping for solutions on the Internet is strongly discouraged. But if you accidentally stumble across it, then it will be accepted as long as you cite the source and explain the solution in your own words.

Homework 5, due November 10

1. Let A_i ($i \in \mathbb{Z}_7$) be finite sets such that for any $i \in \mathbb{Z}_7$ we have $|A_i| = 100$, $|A_i \cap A_{i+1}| = |A_i \cap A_{i+2}| = 10$, $|A_i \cap A_{i+3}| = 0$ and $|A_i \cap A_{i+1} \cap A_{i+2}| = 1$ (all additions are mod 7). Compute, as a plain integer, $\left|\bigcup_{i\in\mathbb{Z}_7} A_i\right|$.

2. How many ways of distributing the standard 32 chess pieces into four distinguishable pouches are there? Pieces of the same kind and color (like two black rooks) are considered indistinguishable; the pouches are allowed to be empty.

   Note: the answer must be a plain integer, but once you come with a closed form expression, you may evaluate it using a calculator without further explanations.

3. Let $\widetilde{p}_n(m)$ be the number of integer partitions of m using at most n numbers in which at least two components are equal.

   (a) Prove that
   $$p_n(m+1) \le p_n(m) + \widetilde{p}_n(m+1)$$
   for any positive integers m, n.

(b) Prove that whenever $m \geq n \geq 3$, this inequality is strict.

4. For a positive integer n, let $A \in \binom{[2n+1]}{n+1}$. Prove that there exist two distinct elements in A that are relatively prime.

5. Give an example of 2022 events $E_1, \ldots, E_{2022}$ in the same sample space such that E_1 and E_{2022} are not independent while for any other pair $1 \leq i < j \leq 2022$, E_i and E_j are independent.