Honors Discrete Mathematics

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Prove all of your answers with reasonable degree of mathematical rigor (feel free to ask us when in doubt). If you work with others put their names clearly at the top of the assignment, everyone must turn in their own independently written solutions. Shopping for solutions on the Internet is strongly discouraged. But if you accidentally stumble across it, then it will be accepted as long as you cite the source and explain the solution in your own words.

Homework 6, due November 17

1. You and $2n$ other people are forming two queues. Everyone (including you) chooses each of the queues with probability 1/2, independently of each other. At the end, one of the queues will be strictly longer than the other. Compute, as a closed form expression, the probability that you are a part of that longer queue.

2. Let X be a *non-negative* random variable. Prove that

$$E(X) \geq \sum_{k=0}^{\infty} 2^{k-1} p(X \geq 2^k).$$

   Note. If you feel uncomfortable with infinite series, you may assume here that X is bounded.

3. Let X be the outcome of a roll of a fair dodecahedral (12 faces, numbered 1 to 12) die. Let Y be the outcome of a roll of a fair icosahedral (20 faces) die. Which of the two quantities $\frac{E(X^3)}{E(X)^2}$, $\frac{E(Y^3)}{E(Y)^2}$ is larger?

4. Prove that $V(X) \leq (c-a)(b-c)$ for any random variable $X \in [a,b]$ with $E(X) = c$.

5. Consider the random variable $i(\sigma)$, where $\sigma \in S_n$ is a random permutation of $[n]$ and $i(\sigma)$ is its *inversion number* defined as the number of pairs $1 \leq i < j \leq n$ for which $\sigma(i) > \sigma(j)$.

   (a) Compute the variance of this random variable as a closed-form expression.

   (b) Prove that
   $$p(i(\sigma) \geq n^2/3) \leq C \cdot n^{-1},$$
   where $C > 0$ is an absolute constant.