Honors Discrete Mathematics

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  Prove all of your answers with reasonable degree of mathematical
rigor (feel free to ask us when in doubt).  If you work with others
put their names clearly at the top of the assignment, everyone must
turn in their own independently written solutions.  Shopping for
solutions on the Internet is strongly discouraged.  But if you accidentally
stumble across it, then it will be accepted as long as you cite
the source and explain the solution in your own words.
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Homework 7, due December 2

1. Assume that $d_1, \ldots, d_n$ is the degree sequence of a simple graph. Prove that the following sequences are also degree sequences of a simple graph (order does not matter):

   (a) $d_1 + 1, d_1 + 1, d_2, d_2, d_3, d_3, \ldots, d_n, d_n$;

   (b) $2d_1, d_2, d_2, d_3, d_3, \ldots, d_n, d_n$;

   (c) $k, d_1 + 1, d_2 + 1, \ldots, d_k + 1, d_{k+1}, \ldots, d_n$;

   (d) $n - 1 - d_1, n - 1 - d_2, \ldots, n - 1 - d_n$.

2. Consider the (undirected simple) *rook graph*: its vertices are the squares of the standard (8×8) chess board, and two squares are adjacent iff the rook can move from one to another.

   Give a closed-form expression for the number of (not necessarily simple) (A1, H8)-paths of length n in this graph.

3. Give an example of two simple graphs that:

(a) have the same degree sequences;

(b) for any given $r \geq 2$ have the same number of copies of K_r;

(c) for any given $\ell \geq 3$ have the same number of induced copies of C_ℓ

but nonetheless are not isomorphic to each other.

4. Prove that for any graph G on n vertices, $\alpha(G)\omega(G) \leq \frac{(n+1)^2}{4}$.

5. Let $G = (V, E)$ be a simple graph. Let $X \subseteq V$ be picked uniformly at random. Prove that

$$E\left(\chi(G|_X)\right) \geq \frac{1}{2}\chi(G).$$