Honors Discrete Mathematics

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Prove all of your answers with reasonable degree of mathematical rigor (feel free to ask us when in doubt). If you work with others put their names clearly at the top of the assignment, everyone must turn in their own independently written solutions. Shopping for solutions on the Internet is strongly discouraged. But if you accidentally stumble across it, then it will be accepted as long as you cite the source and explain the solution in your own words.

Homework 1, due October 11

1. Let $a_n \stackrel{\text{def}}{=} \sqrt{1 \cdot 2} + \sqrt{2 \cdot 3} + \ldots + \sqrt{n(n+1)}$ $(n \geq 1)$. Prove that
$$\frac{1}{2}n(n+1) < a_n < \frac{1}{2}(n+1)^2.$$

2. A function $f : \mathbb{N} \longrightarrow \mathbb{N}$ satisfies the following two properties:
$$f(n) = n - 7 \quad \text{if} \quad n > 49$$
$$f(n) = f(f(n+8)) \quad \text{if} \quad n \leq 49.$$
Compute $f(9)$.

3. Prove or disprove the following. Let $f, g, f', g' : \mathbb{N} \longrightarrow \mathbb{N}$ be functions such that $f(n) \leq O(g(n))$, $f'(n) \leq O(g'(n))$, $\forall n(f(n) > f'(n))$ and $\forall n(g(n) > g'(n))$. Then $f(n) - f'(n) \leq O(g(n) - g'(n))$.

4. For a linear (or partial) ordering $(S, \leq)$, $(S, \leq^\top)$ is its *dual* obtained from $(S, \leq)$ by reversing directions: $a \leq^\top b$ iff $b \leq a$. Which of the following ordered sets:
$$(\mathbb{N}, \leq),\ (\mathbb{Q}, \leq),\ (\mathbb{R}_{>0}, \leq),\ (\mathbb{R}_{\geq 0}, \leq)$$

are *self-dual*, that is isomorphic to their own duals (and why)?

5. Prove that there does not exist any partial ordering of [4] that has precisely 17 different total extensions.