Honors Discrete Mathematics

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`Prove all of your answers with reasonable degree of mathematical rigor (feel free to ask us when in doubt). If you work with others put their names clearly at the top of the assignment, everyone must turn in their own independently written solutions. Shopping for solutions on the Internet is strongly discouraged. But if you accidentally stumble across it, then it will be accepted as long as you cite the source and explain the solution in your own words.`

Homework 2, due October 18

1. We saw in class that $\omega + 1 \neq \omega$. Is it also true that $(\omega + 1) \cdot \omega \neq \omega^2$?

2. Let $(S, \leq)$ be a well-ordered set and $f : S \longrightarrow S$ be a function such that for all $x, y \in S$, $f(x) \leq f(y)$ if and only if $x \leq y$. Prove that $f(x) \geq x$, again for all $x \in S$.

3. Compute gcd(x,y) where (x, y) is the output of `Bezout`(a, b) for some $a > b > 0$.

4. Generalizing the case $k = 2$, $\gcd(a_1, \ldots, a_k)$ is defined as the largest positive d such that $d|a_i$ for all $i \in [k]$. Let $(a_{ij} \mid i \in [n],\ j \in [m])$ be an $n \times m$ matrix consisting of positive integers. Prove the inequality

$$\prod_{j=1}^{m} \gcd(a_{1j}, a_{2j}, \ldots, a_{nj}) \leq \gcd\left(\prod_{j=1}^{m} a_{1j},\ \prod_{j=1}^{m} a_{2j},\ \ldots, \prod_{j=1}^{m} a_{nj}\right).$$

5. Prove that

$$\left\{\gcd(n^2 + n + 1,\ n^2 - 1) \mid n \in \mathbb{N}\right\} = \{1, 3\}.$$