Honors Discrete Mathematics

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`Prove all of your answers with reasonable degree of mathematical rigor (feel free to ask us when in doubt). If you work with others put their names clearly at the top of the assignment, everyone must turn in their own independently written solutions. Shopping for solutions on the Internet is strongly discouraged. But if you accidentally stumble across it, then it will be accepted as long as you cite the source and explain the solution in your own words.`

Homework 3, due October 25

1. Consider the structure $(\mathbb{R}, \oplus, \otimes)$ in which "addition" $\oplus$ and "multiplication" $\otimes$ are defined as follows:

$$x \oplus y \stackrel{\text{def}}{=} \min(x, y)$$
$$x \otimes y \stackrel{\text{def}}{=} x + y.$$

   What axioms of a field does this structure satisfy? (The first thing to understand is whether it contains neutral elements w.r.t. addition and multiplication, aka "0" and "1".)

2. Consider the following binary relation $R(a, b)$ on $\mathbb{R}$: $R(a, b)$ if and only if $a^{10} - b^{10} = a - b$. Prove that this is an equivalence relation and that[1] every equivalence class has ≤ 2 elements.

[1] This part may require a bit of very simple analysis.

3. Let R be a commutative ring and $\mathcal{I}_R$ be the set of all its ideals, including trivial ideals $\{0\}, R$. For $I, J \in \mathcal{I}_R$, let

$$\begin{aligned} I + J &\stackrel{\text{def}}{=} \{a + b \mid a \in I,\ b \in J\} \\ I \cdot J &\stackrel{\text{def}}{=} \{a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \mid n \geq 0,\ a_1, \ldots, a_n \in I,\ b_1, \ldots, b_n \in J\}. \end{aligned}$$

   (a) Verify that $I + J, I \cdot J \in \mathcal{I}_R$.

   (b) Which of the following hold for all commutative rings R and any $I, J, K \in \mathcal{I}_R$:

      i. $I \cdot (J + K) = IJ + IK$;

      ii. $(I + K) \cdot (J + K) \subseteq IJ + K$;

      iii. $IJ + K \subseteq (I + K) \cdot (J + K)$?

4. Let p and q be two distinct primes. How many solutions in $\mathbb{Z}_{pq}$ does the congruence $x^2 \equiv 3x \mod pq$ have?

   `Note. The answer may depend on` p, q.

5. Give a closed form expression for the number of ordered triples $\langle A, B, C\rangle$ such that $A, B, C \subseteq [n]$ and $A \cap B \subseteq C \subseteq A \cup B$.