Honors Discrete Mathematics

Instructor: Alexander Razborov, University of Chicago
razborov@uchicago.edu
Course Homepage: people.cs.uchicago.edu/˜razborov/teaching/autumn23.html

Autumn Quarter, 2023

`Prove all of your answers with reasonable degree of mathematical rigor (feel free to ask us when in doubt). If you work with others put their names clearly at the top of the assignment, everyone must turn in their own independently written solutions. Shopping for solutions on the Internet is strongly discouraged. But if you accidentally stumble across it, then it will be accepted as long as you cite the source and explain the solution in your own words.`

Homework 4, due November 3

1. Let us call a function $f : X \longrightarrow Y$ *gorgeous* if for any function $g : Z \longrightarrow Y$ with $|Z| \leq 17$ there exists an injective function $h : Z \rightarrowtail X$ such that $g = f \circ h$. Let us call f *co-gorgeous* if for any $g : X \longrightarrow Z$, again with $|Z| \leq 17$, there exists a surjective $h : Y \twoheadrightarrow Z$ such that $g = h \circ f$.

   (a) Describe the set of those integers n for which there exists a gorgeous function $f : [100] \longrightarrow [n]$.

   (b) Describe the set of those integers n for which there exists a co-gorgeous function $f : [100] \longrightarrow [n]$.

2. How many ways are there to choose six cards out of a standard deck of 52 cards in such a way that each suit is represented by either one or two cards in the collection? `Note. For full credit, the answer must be in the form of a plain integer.`

3. (a) Show that for $k \leq n$ we have

$$\sum_{i=0}^{k} (-1)^i \binom{n}{i} \geq 0$$

if k is even and

$$\sum_{i=0}^{k} (-1)^i \binom{n}{i} \leq 0$$

if k is odd.

(b) Use this to verify the statement made in class: partial sums in the inclusion-exclusion expression oscillate around the actual value.

4. Prove that

$$S(m_1 + m_2, n) \geq n! S(m_1, n) S(m_2, n),$$

for all positive integers n, m_1, m_2.

5. Let A_i ($i \in \mathbb{Z}_6$) be sets such that for any $i \in \mathbb{Z}_6$ we have $|A_i| = 50$, $|A_i \cap A_{i+1}| = 20$, $A_i \cap A_{i+2} = \emptyset$ and $|A_i \cap A_{i+3}| = 5$, where all additions are $\mod 6$. Compute $|A_0 \cup \ldots \cup A_5|$.