Honors Discrete Mathematics

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   Prove all of your answers with reasonable degree of mathematical
rigor (feel free to ask us when in doubt).  If you work with others
put their names clearly at the top of the assignment, everyone must
turn in their own independently written solutions.  Shopping for
solutions on the Internet is strongly discouraged.  But if you accidentally
stumble across it, then it will be accepted as long as you cite
the source and explain the solution in your own words.
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Homework 6, due November 15

1. Let $k \leq n/2$ be positive integers. Construct n events $(E_i \mid i \in \mathbb{Z}_n)$ in the same sample space such that for any $i \in \mathbb{Z}_n$ the events[1] $E_i, E_{i+1}, \ldots E_{i+k-1}$ are mutually independent but E_i and E_{i+k} are not independent, again for any $i \in \mathbb{Z}_n$.

2. A fair six-sided die is rolled n times. Give a closed form expression for the probability that the *product* of all n outcomes is divisible by 15.

3. Alice plays a slot machine with probability of winning p in each round until she wins 10 times and then she goes home. Compute the distribution of the number of plays as a closed form expression.

4. Recall that the *mean deviation* $MD(X)$ of a random variable X is defined as $E(|X - c|)$, where $c = E(X)$ is the expectation of X.

   (a) Prove that for any two random variables X and Y on the same sample space, $MD(X + Y) \leq MD(X) + MD(Y)$.

[1] the summation in indices is $\mod n$

(b) Prove that if X and Y are additionally known to be independent, then this inequality is *always* strict, unless one of the variables X, Y is constant (that is, takes one fixed value with probability 1).

5. Let $X_1, \dots, X_n$ be *pairwise* independent random variables such that $X_1 + \cdots + X_n = 1$ (with probability 1). Prove that they all must be constant.