Honors Discrete Mathematics

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   Prove all of your answers with reasonable degree of mathematical
rigor (feel free to ask us when in doubt).  If you work with others
put their names clearly at the top of the assignment, everyone must
turn in their own independently written solutions.  Shopping for
solutions on the Internet is strongly discouraged.  But if you accidentally
stumble across it, then it will be accepted as long as you cite
the source and explain the solution in your own words.
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Homework 7, due December 1

1. Assume that $d_1, \ldots, d_n$ is the degree sequence of a simple graph. Prove that the following sequences are also degree sequences of a simple graph (order does not matter):

   (a) $d_1 + 1, d_1 + 1, d_2, d_2, d_3, d_3, \ldots, d_n, d_n$;

   (b) $k, d_1 + 1, d_2 + 1, \ldots, d_k + 1, d_{k+1}, \ldots, d_n$;

   (c) $n - 1 - d_1, n - 1 - d_2, \ldots, n - 1 - d_n$.

2. Let G be a triangle-free Δ-regular graph on n vertices such that every two non-adjacent vertices have precisely two common neighbours. Compute, as a closed form expression, the number of cycles of length 4 in this graph. Order and orientation do not matter: a cycle is identified with the set of its (non-oriented) edges.

3. Consider the path graph P_{2N+1} on the set of vertices $\{-N, -N + 1, \ldots, -1, 0, 1, \ldots, N - 1, N\}$: u and v are connected if and only if $|u - v| = 1$. Let $n \leq N$. Compute, as a closed form expression, the number of $(0, n)$-paths (*not* necessarily simple!) of length precisely N in this graph.

4. Give an example of two simple graphs that:

   (a) have the same degree sequences;

   (b) for any given $r \geq 2$ have the same number of copies of K_r;

   (c) for any given $\ell \geq 3$ have the same number of induced copies of C_ℓ

   but nonetheless are not isomorphic to each other.

5. Compute, as a closed form expression, $\mathrm{ex}(n; K_{n-1}^{(n-2)})$. In words: what is the maximum number of hyperedges an $(n-2)$-hypergraph on n vertices can have if it is known that no $n-1$ vertices span a clique?