Discrete Mathematics

Instructor: Alexander Razborov, University of Chicago
razborov@cs.uchicago.edu
Course Homepage: www.cs.uchicago.edu/~razborov/teaching/autumn10.html

Autumn Quarter, 2010

`Prove all of your answers. If you work with others put their names clearly at the top of the assignment. Everyone must turn in their own independently written solutions. Homework is due at the beginning of class.`

Homework 2, due October 20

1. For simplicity, all functions considered in this exercise are from the set of positive integers $\{1, 2, \ldots, n, \ldots\}$ to itself. We say that $f(n) \leq o(g(n))$ if for any $\epsilon > 0$ there exists $k > 0$ such that for all $n \geq k$ we have $f(n) \leq \epsilon g(n)$ (equivalently, $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$).

   Prove or disprove the following:

   (a) If $f_1(n) \leq O(g_1(n))$ and $f_2(n) \leq O(g_2(n))$ then $f_1(n)^{f_2(n)} \leq O(g_1(n)^{g_2(n)})$;

   (b) If $f_1(n) \leq o(g_1(n))$ and $f_2(n) \leq o(g_2(n))$ then $f_1(n)^{f_2(n)} \leq o(g_1(n)^{g_2(n)})$.

2. We have a set of 1024 banknotes one of which is counterfeit, and we are given a device that, given any set of banknotes, can detect if this set contains a counterfeit banknote or not (but, in case of success, the device does not tell you which one is counterfeit). How many applications of this device does one need to find the counterfeit banknote in the worst case? Prove both upper and lower bounds, i.e., give an algorithm and prove that no other algorithm with smaller number of applications can always guarantee success.

3. Let us denote by $\mathrm{ic}(a_1, \ldots, a_n)$ the number of inversions in a list of integers $a_1, \ldots, a_n$, and, for a sorting algorithm A, let $t(A; a_1, \ldots, a_n)$ be the overall number of *comparisons* made by A on the input $(a_1, \ldots, a_n)$ (thus, now we also charge for comparing a_i and a_j even if they turn out to be in the right order).

   (a) Prove that $t(\mathrm{InsertionSort}; a_1, \ldots, a_n) \leq \mathrm{ic}(a_1, \ldots, a_n) + n$;

   (b) Does there exist a sorting algorithm A such that $t(A; a_1, \ldots, a_n) \leq \mathrm{ic}(a_1, \ldots, a_n) + \sqrt{n}$ for any input $(a_1, \ldots, a_n)$?

4. Prove that a graph is bipartite iff it contains no odd cycles.

5. Suppose we color the edges of K_6, a complete graph on 6 vertices, with two colors. Prove that no matter how we color the edges there will exist a monochromatic triangle (that is, a triangle in which all edges have the same color).