Discrete Mathematics

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Autumn Quarter, 2010

`Prove all of your answers. If you work with others put their names clearly at the top of the assignment. Everyone must turn in their own independently written solutions. Homework is due at the beginning of class. Large factorials` $n!$ `with` $n \geq 7$ `may be left unexpanded.`

Homework 7, due November 24

1. Let X be a *non-negative* random variable. Prove that

$$E(X) \geq \sum_{k=0}^{\infty} 2^{k-1} p(X \geq 2^k).$$

   `Note. If you feel uncomfortable with infinite series, you may assume here that` X `takes on only finitely many values with non-zero probability.`

2. We roll a pair of fair dice. Compute the expectation of the *ratio* of the number of points rolled on the first die to the number of points rolled on the second.

3. The *mean deviation* $MD(X)$ of a random variable X is defined as $E(|X - c|)$, where $c = E(X)$ is the expectation of X (we briefly discussed this notion in the class, and denounced it).

   (a) Prove that for any two random variables X and Y on the same sample space, $MD(X + Y) \leq MD(X) + MD(Y)$.

(b) Prove that if X and Y are additionally known to be independent, then this inequality is *always* strict, unless one of the variables X, Y is trivial (that is, takes one fixed value with probability 1).

4. The probability that a coin comes up head is p. The coin is flipped repeatedly until the first time we see the combination HT (that is, head *immediately* followed by tail). Calculate the distribution of the number of flips, i.e. give a closed form expression for the probability that the number of flips is equal to k for any integer $k \geq 2$. `Note. If you find this problem too easy, you can try to do (not for credit!) its version with the HH combination.`

5. Let $\sigma : [n] \longrightarrow [n]$ be picked uniformly at random from the set of all permutations on n elements. Recall that $\mathrm{ic}(\sigma)$ is the inversion complexity of σ. Prove that

$$p\left(\mathrm{ic}(\sigma) \geq \frac{n^2}{3}\right) \leq \frac{3}{4}.$$