Discrete Mathematics

Instructor: Alexander Razborov, University of Chicago
razborov@cs.uchicago.edu
Course Homepage: www.cs.uchicago.edu/~razborov/teaching/autumn13.html

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Prove all of your answers.  If you work with others put their
names clearly at the top of the assignment.  Everyone must turn
in their own independently written solutions.  Homework is due at
the beginning of class.
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Homework 1, due October 16

1. Find polynomials $x(n), y(n)$ such that

$$x(n)(n^2 + n + 1) + y(n)(n^2 + 1) = 1$$

   for all integer n.

2. Similarly to the case $k = 2$, define the *greatest common divisor* $\gcd(a_1, \ldots, a_k)$ of the integers $a_1, \ldots, a_k$ as the largest integer d such that $d|a_1, \ldots, d|a_k$.

   Give an example of integers a, b, c, d such that $\gcd(u, v, w) = 1$ for any three-element subset $\{u, v, w\} \subseteq \{a, b, c, d\}$, while $\gcd(u, v) > 1$ for any 2-element subset.

3. Recall that a *graph* is a collection of *vertices* in which some pairs are connected by an *edge*, and a *perfect matching* in a graph with an even number of vertices n is a set of $n/2$ edges pairing its vertices.

   Which of the two graphs on Figure 1 have a perfect matching? Exhibit the matching if you think the answer is positive, and explain why if it is negative.

4. $2n$ football players each of whom has an odd number of friends among the others are distributed into two teams. A player is *happy* if he has

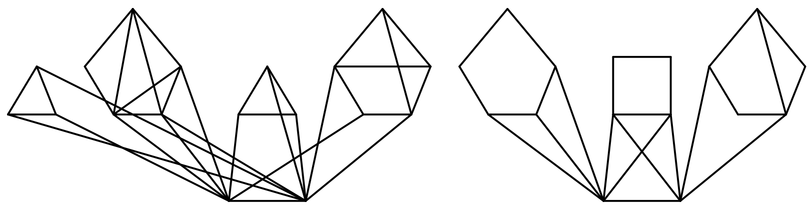

Figure 1: Two similar graphs

more friends on the other side than on his own, and the distribution is *stable* if everyone is happy. The *cut number* of a distribution is the number of pairs of friends belonging to the opposite teams.

Give an example of two different **stable** distributions of the same set of players that have different cut numbers.

5. Prove that a function $f : X \longrightarrow Y$ is surjective if and only if for any finite set Z and any function $g : Z \longrightarrow Y$ there exists a function $h : Z \longrightarrow X$ such that g is their composition: $g = f \circ h$.