Discrete Mathematics

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Prove all of your answers. If you work with others put their
names clearly at the top of the assignment. Everyone must turn
in their own independently written solutions. Homework is due at
the beginning of class.
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Homework 5, due November 14

1. A fair (unbalanced) coin is tossed repeatedly until the first time we see the combination HEAD TAIL (in this order, but *not* necessarily consecutively). Calculate the distribution of the number of tosses, i.e. give a close form expression for the probability that their number in this experiment is equal to k for any integer $k \geq 2$.

2. You roll three fair dice. Compute the expectation of the *maximum* of the three results, and the expectation of their *minimum*.

3. For a function $f : [n] \longrightarrow [n]$, let us define its *inverse* complexity $\mathrm{ic}(f)$ as the number of ordered pairs $\langle i, j \rangle$ such that $i < j$ and $f(j) \leq f(i)$.

   Prove that
$$p\left(\mathrm{ic}(f) \geq \frac{n^2}{3}\right) \leq \frac{3}{4},$$
   where f is chosen uniformly at random from the set of all functions.

4. The *mean deviation* $MD(X)$ of a random variable X is defined as $E(|X - c|)$, where $c = E(X)$ is the expectation of X (we briefly discussed this notion in class, and demoted it).

(a) Prove that for any two random variables X and Y on the same sample space, $MD(X+Y) \leq MD(X) + MD(Y)$.

(b) Prove that if X and Y are additionally known to be independent, then this inequality is *always* strict, unless one of the variables X, Y is trivial (that is, takes one fixed value with probability 1).

5. Let again f be picked uniformly at random from the set of all functions from $[n]$ to $[n]$. Give a close form expression for the variance of the random variable $|\mathrm{im}(f)|$.