Quantum Computing

Instructor: Alexander Razborov, University of Chicago.
razborov@cs.uchicago.edu

Course Homepage: www.cs.uchicago.edu/~razborov/teaching/autumn15.html

Autumn Quarter, 2015

Homework 3, due December 1

1. By analogy with communication complexity, let us define the composition of two total functions\( g : \{0, 1\}^m \to \{0, 1\}, \ h : \{0, 1\}^n \to \{0, 1\} \) by
\[
(g \circ h)(x^1, \ldots, x^m) \overset{\text{def}}{=} g(h(x^1), \ldots, h(x^m)),
\]
where all \( mn \) variables \( x^1, \ldots, x^m \) are pairwise different.

(a) Prove that \( C(g \circ h) \leq C(g) \cdot C(h) \), where \( C(f) \) is the certificate complexity of \( f \).

(b) Is this bound always tight?

2. Let
\[
f(x, y) = \begin{cases} 
1 & \text{iff } x \leq y \leq x + 2015 \\
0 & \text{otherwise}
\end{cases}
\]
(we view \( x \) and \( y \) as integers in \( \{0, 1, \ldots, 2^n - 1\} \)). Prove that the randomized communication complexity of this function is \( O(\log n) \).

3. Consider the following statement \( P(k) \) (\( k \geq 1 \) an integer):

Let \( n \geq 1 \) be an integer, and let \( (p_1, |\phi_1\rangle), \ldots, (p_k, |\phi_k\rangle) \) and \( (q_1, |\psi_1\rangle), \ldots, (q_n, |\psi_n\rangle) \) be two mixed states in the same Hilbert space that have the same density matrix. Then for every \( 1 \leq i \leq k \) there exists \( 1 \leq j \leq n \) and a real \( \theta \) such that \( |\phi_i\rangle = e^{i\theta} |\psi_j\rangle \) (i.e., these two states differ only by a global phase).

Describe those \( k \) for which \( P(k) \) is true.
4. Compute explicit representation of the one-qubit depolarizing channel
\( \mathcal{E}_\eta(\rho) = (1 - \eta)\rho + \frac{\eta}{2} I \) in the operator form \( \sum_k p_k (U_k \rho U_k^\dagger) \), where \( U_k \in \{I, X, Y, Z\} \) are Pauli matrices.