Computability and Complexity Theory

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   You are encouraged to work together on solving homework problems,
but please put their names clearly at the top of the assignment.
Everyone must turn in their own independently written solutions.
Homework is due at the beginning of class.
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Homework 1, due February 10

1. The *Ackermann function* $A(m,n)$ is given by the recursion

$$\begin{cases} A(0,n) \stackrel{\text{def}}{=} n+1 \\ A(m+1,0) \stackrel{\text{def}}{=} A(m,1) \\ A(m+1,n+1) \stackrel{\text{def}}{=} A(m, A(m+1,n)). \end{cases}$$

   Find all non-negative integer solutions of the equation $A(m,n) = m + n$.

2. Design a Markov algorithm for computing the function $f(x) = \lfloor x/2 \rfloor$.

3. An URM M *oddly accepts* an input x if in the course of its computation on x, 0 appears in the register R_1 only finitely many times, and, moreover, this number is odd. Let

$$L_{\text{odd}} \stackrel{\text{def}}{=} \{\, (x,y) \mid M_x \text{ oddly accepts } y \,\}$$

   (M_x is the URM with code x).

   (a) Prove that every r.e. set is many-one reducible to L_{odd}.

(b) Prove that every r.e. set is many-one reducible to its complement $co - L_{\text{odd}}$.

(c) Prove that L_{odd} is not r.e.

4. Prove that the intersection of two simple sets is simple.

5. Let G be a *recursive graph* on $\mathbb{N}$, i.e. a recursive subset $E = E(G) \subseteq \mathbb{N} \times \mathbb{N}$ such that $\forall x \in \mathbb{N}(\langle x, x\rangle \notin E)$ and $\forall x, y \in \mathbb{N}(\langle x, y\rangle \in E \equiv \langle y, x\rangle \in E)$. Recall that $Z(x)$ is the identically zero function in one argument.

   (a) Which of the following operators on $\mathcal{F}_2$:

$$\Phi_0(f)(x,y) \stackrel{\text{def}}{=} \begin{cases} 0, & \text{if } (x,y) \in E \\ \mu z(f(x,z) + f(y,z) \geq 0) & \text{otherwise} \end{cases}$$

$$\Phi_1(f)(x,y) \stackrel{\text{def}}{=} \begin{cases} 0, & \text{if } (x,y) \in E \\ Z(\mu z(f(x,z) + f(y,z) \geq 0)) & \text{otherwise} \end{cases}$$

   is recursive and why?

   (b) Compute the least fixed point of this operator (the answer should be given in graph-theoretical terms).