Computability and Complexity Theory

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Winter Quarter, 2017

`You may work together on solving homework problems, but please put all the names clearly at the top of the assignment. Everyone must turn in their own independently written solutions. If you find any of the solutions on-line, it is OK but then you must completely understand the proof, explain it in your own words and include the URL. The due date below pertains to e-mail submissions as a PDF file prepared from a TeX source.`

Homework 2, due February 24

1. It is a well-known observation (particularly in the math recreational literature) that in every non-empty bar there always exists a customer that can rightfully shout "When I drink, everybody drinks". In the first-order logic this is expressed by the sentence

$$\exists x(x = x) \Rightarrow \exists x(D(x) \Rightarrow \forall y D(y)).$$

   Give a formal (list all axioms and inference rules you are using) proof of this sentence in the first-order calculus. You may use the version from any textbook in existence, but if your axioms/rules look suspiciously exotic, please supply a reference.

2. (a) Show that the inequality is definable over $\mathbb{Z}$ in the language $L = \langle 0, +, x^2 \rangle$, that is there exists a first-order formula $M(x, y)$ in this language such that for arbitrary $\mathbf{m}, \mathbf{n} \in \mathbb{Z}$, $\mathbf{m} \leq \mathbf{n}$ if and only if $\mathbb{Z} \models M(\mathbf{m}, \mathbf{n})$.

   (b) Is it definable (in the same sense) in the language $L = \langle 0, +, x^3 \rangle$?

3. Prove that there is no algorithm that for any given system

$$p_1(x_1, \ldots, x_n) = 0, \ldots, p_m(x_1, \ldots, x_n) = 0$$

of quadratic (i.e. of degree ≤ 2) polynomial equations tells if the system has an integer solution or not. You may refer to Matiyasevich's theorem, of course.

4. Prove that

$$K(x_0, \ldots, x_4) \leq \frac{1}{2} \sum_{i \in \mathbb{Z}_5} K(x_i, x_{i+1}) + O(\log K(x_0, \ldots, x_4)).$$

5. Let $\omega_1 \omega_2 \ldots \omega_n \ldots$ be a (Martin-Löf) random string. Prove that $\{\, i \in \mathbb{N} \mid \omega_i = 1 \,\}$ is immune.