Computability and Complexity Theory

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You are encouraged to work together on solving homework problems,
but please put their names clearly at the top of the assignment.
Everyone must turn in their own independently written solutions.
Homework is due at the beginning of class.
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Homework 1, due February 9

1. The *Ackermann function* $A(m,n)$ is given by the recursion

$$\begin{cases} A(0,n) \stackrel{\text{def}}{=} n+1 \\ A(m+1,0) \stackrel{\text{def}}{=} A(m,1) \\ A(m+1,n+1) \stackrel{\text{def}}{=} A(m, A(m+1,n)). \end{cases}$$

   Prove that it is strictly increasing in both its arguments.

2. An URM M *ω-accepts* an input x if in the course of its computation on x, 0 appears infinitely often in the register R_1 (in particular, M does not stop on x). Let

$$L_\omega \stackrel{\text{def}}{=} \{\, (x,y) \mid \phi_x \ \omega\text{-accepts } y \,\}.$$

   (a) Prove that every r.e. set is many-one reducible to L_ω.

   (b) Prove that every r.e. set is many-one reducible to its complement $co - L_\omega$.

   (c) Prove that L_ω is not r.e.

3. Prove that there exist two r.e. sets L_1 and L_2 such that *neither* of the four sets $L_1 \cap L_2$, $L_1 \cup L_2$, $L_1 \setminus L_2$, $L_2 \setminus L_1$ is recursive.

4. It is a well-known observation that in every non-empty bar there always is a customer that can rightfully shout "When I drink, everybody drinks". In the first-order logic this is expressed by the sentence

$$\exists x(x = x) \supset \exists x(D(x) \supset \forall y D(y)).$$

Give a formal (i.e., list all axioms and inference rules you are using) proof of this sentence in the first-order calculus. You may use the version from any existing book, but if your axioms/rules look exotic/complicated, please supply a reference.

5. (a) Prove that for any first-order formula A in the first-order language $\langle 0, S, \leq \rangle$ (possibly with free variables) there exists another **open** formula B such that the equivalence $A \equiv B$ is true on the set of all non-negative integer numbers.

   (b) Does a similar statement hold for the language $\langle 0, \leq \rangle$?