The Dimension of the Space of C^1 Piecewise Polynomials*

John Morgan
Department of Mathematics
Columbia University

Ridgway Scott
Department of Mathematics
University of Houston

March 26, 1996

Abstract

We present a method for computing the dimension of C^1 piecewise polynomials on a triangulated polygonal domain in the plane. Our results verify a conjecture of Strang in a large number of cases. For fourth–degree piecewise polynomials we define, inductively, a nodal basis on quite general meshes.
AMS/MOS Numbers: 65D07, 65N30.

1 Introduction and Strang's Conjecture

Let Π be a triangulated polygon in the plane, $\mathbf{R}^2$. We say a function ϕ is a C^1 *piecewise polynomial of degree* n in Π if $\phi \in C^1(\Pi)$ and ϕ restricted to each triangle is a polynomial of degree $\leq n$. We denote by $S_n(\Pi, \mathcal{T})$ the space of such functions, where $\mathcal{T}$ refers to the triangulation. In an earlier note [9], we described the structure of S_n completely for $n \geq 5$ by exhibiting a nodal basis answering the conjecture of Strang [14, 15]. In this paper, we continue the study of Strang's conjecture by calculating the dimension of S_n for $n < 5$.

The importance of S_n lies in diverse areas. Perhaps the most straightforwards is in its use in approximating solutions of differential equations via variational techniques [8, 16]. For example, the function u that minimizes

$$I(v) = \int_\Pi v_{xx}^2 + 2v_{xy}^2 + v_{yy}^2 + v^2 - 2fv \; dxdy \tag{1.1}$$

satisfies the differential equation

$$\Delta^2 u + u = f \; in \; \Pi$$

together with some "natural" boundary conditions (see [2], p. 235). If $\phi \in S_n$, then we may view the second derivatives of ϕ as well defined in $L^\infty(\Pi)$, so the functional I in (1.1) is well defined on S_n. The variational approximation, u^*, to u is the function in S_n that minimizes I.

*This work was supported in part by the National Science Foundation through award number DMS-8903548.

There is a natural norm associated with I, namely,

$$\| v \|^2 = \int_\Pi v_{xx}^2 + 2v_{xy}^2 + v_{yy}^2 + v^2 dxdy$$

and the fundamental result of variational approximation is that u^* is the projection of u onto S_n with respect to this norm:

$$\| u - u^* \| = \inf_{v \in S_n} \| u - v \| .$$

Thus, error estimates for $u - u^*$ are reduced to an approximation problem. We have not studied this problem in general, but the results of [9] show that for $n \geq 5$ the solution follows familiar lines (see, e.g., [8]). Approximation properties (with respect to mesh refinement) have been shown to be sub–optimal by de Boor, Höllig *et al.* (cf. [6]) for $n \leq 4$, but some results of Powell [10] show that on a special mesh ($\mathcal{T}_2$ of Section 3) even S_2 approximates well.

The determination of u^* in the context above requires a basis for S_n, which we have not found for $n \leq 3$. However in [11] the second author studied a technique for determining u^* that does not require an explicit basis for S_n. The results there rely on knowing the dimension of S_n.

A more subtle application of S_n arises in finite element methods for problems with incompressibility constraints, e.g., the Stokes equation of fluid flow [13]. In this case, S_n is isomorphic to the space of divergence–zero vector fields that are continuous, piecewise polynomials of degree $n-1$ (on the same mesh); the isomorphism is provided by the curl ($\bigtriangledown\times$) operator [12].

Finally, C^1 piecewise polynomials (and piecewise polynomials of even greater smoothness) are of interest in computer graphics for the representation of discrete data, cf. [10] and references in [1].

We now recall the heuristic method used by Strang [14, 15] to compute dim S_n. We view S_n as the space of discontinuous piecewise polynomials of degree $\leq n$ ($\simeq (\mathcal{P}_n)^T$, where $\mathcal{P}_n$ is the space of polynomials of degree $\leq n$ and T is the number of triangles in $\mathcal{T}$) with constraints imposed across each interior edge in $\mathcal{T}$. If E_0 is the number of interior edges in $\mathcal{T}$, then the number of constraints imposed is $(2n+1)E_0$: $n+1$ for the value and n for the normal derivative on each interior edge. These constraints are redundant, though. Consider an interior vertex and impose constraints on all but one edge meeting there. Then functions satisfying these constraints are already C^1 at the vertex, and only $(2n+1)-3$ more constraints need be imposed on the remaining edge to insure C^1 continuity. Thus we have shown that

$$\dim S_n(\Pi, \mathcal{T}) \geq \frac{1}{2}(n+1)(n+2)T - (2n+1)E_0 + 3V_0, \tag{1.2}$$

where V_0 is the number of interior vertices in $\mathcal{T}$. Strang conjectured that (at least for generic triangulations) equality held in (1.2).

Some work of Powell [10] had already shown that genericity was needed, as follows. Let Π be a convex quadrilateral and triangulate it by drawing in the diagonals. Thus $T = 4$, $E_0 = 4$, and $V_0 = 1$, and the conjectured dimension of $S_2(\Pi, \mathcal{T})$ would be

$$6 \cdot 4 - 5 \cdot 4 + 3 \cdot 1 = 7.$$

In other words, the conjecture is that there is only one *piecewise* quadratic on this triangulation, since $\dim \mathcal{P}_2 = 6$. However, we can construct two that are linearly independent, namely, for each diagonal we let ϕ_i $(i = 1, 2)$ be the function that is zero on one side of the diagonal and equal to (distance from the diagonal squared) on the other side. Since ϕ_i vanishes to second order on the i-th diagonal, it is C^1. ϕ_1 and ϕ_2 are linearly independent modulo $\mathcal{P}_2$ as long as the diagonals are distinct, so we have proved that

$$\dim S_2(\Pi, \mathcal{T}) \geq 8.$$

In Lemma 1, we show that the dimension is exactly 8, and in Lemma 2, we show that the dimension drops to 7 if the interior vertex is moved.

We refer to the above situation above as *singular* (Definition 1), and we prove in a variety of examples that the above example characterizes the only way the dimension of S_n can differ from the minimal dimension in (1.2). Thus we will show (Theorem 1) that

$$\dim S_n(\Pi, \mathcal{T}) \geq \frac{1}{2}(n+1)(n+2)T - (2n+1)E_0 + 3V_0 + \sigma \tag{1.3}$$

where σ is the number of singular vertices in $\mathcal{T}$. In particular, we show that equality holds in (1.3) for $n = 4$ provided the triangulation contains no "quasi–singular" vertices, and moreover we construct a nodal basis for S_4 in this case. These results generalize slightly those of Alfeld, Piper and Schumaker [1] in that we construct a basis that allows interpolation of the value and gradient at all vertices for a larger class of triangulations. It is clear that showing that $\dim S_n$ takes on the minimal dimension in (1.2) is a problem in algebraic geometry, so that if the dimension is minimal for a given $\mathcal{T}$, it will be minimal for an open dense set of perturbations of $\mathcal{T}$. That this is so for $n \geq 2$ has been proved by Billera [3] and Whitely [17].

The plan of the paper is as follows: in the next section (Section 2), we develop a method for computing $\dim S_n$ valid for general $\mathcal{T}$ when $n \geq 2$ (if $n \leq 1$, S_n is trivial). In Section 3, we verify that equality holds in the formula (1.3) for $n \geq 2$ on two special triangulations and show by example that one cannot expect to find (generically) C^1 piecewise quadratics with compact support. Section 4 studies in detail a nonsingular example where the formula of Theorem 1 is invalid for $n = 2$ (i.e., the minimum dimension is not achieved, although we show it is achieved for a generic perturbation of the example). This example shows that although the constraints defining $S_2(\Pi, \mathcal{T})$ are locally independent, they may be globally dependent for some $\mathcal{T}$'s. We also show in Section 5 (Theorem 4) that equality holds in (1.3) for $n = 3$ for a fairly wide class of triangulations, including the one where it fails for $n = 2$. In Section 6, we apply some results of the previous section to prove that the Clough–Toucher and Fraeijs de Veubeke–Sander macro–cubic elements are well defined. In Section 7, we describe a nodal basis for $S_4(\Pi, \mathcal{T})$ in the case that $\mathcal{T}$ contains no "quasi–singular" vertices; in this case, it easily follows that equality holds in (1.3) for $n = 4$.

This paper is a slightly modified version of a manuscript written by the authors in 1975 and circulated informally since that time. The original manuscript had attempted, unsuccessfully, to prove that equality holds in (1.3) for $n = 4$ for all triangulations. However, the treatment of quasi–singular vertices was flawed, and repeated attempts to complete the proof when quasi–singular vertices occur have failed. Due to the continued (and growing) interest in the subject, it was decided to publish a modified version of the original manuscript (limited essentially to triangulations not containing quasi–singular vertices) in order to make the results available to a broader audience.

2 Dimension Counting Via Induction

Let Π be a bounded domain in $\mathbf{R}^2$ such that $\partial\Pi$ consists of a finite number of non–intersecting polygonal arcs. (By a polygonal arc, we mean a continuous curve that is the union of a finite number of line segments.) Let $\mathcal{T}$ be a triangulation of Π.

DEFINITION 1. A vertex v is *singular* [9] in $\mathcal{T}$ if (see Figure 1)

(i) v lies in the interior of Π,

(ii) exactly four edges meet at v and

(iii) the edges form two straight lines.

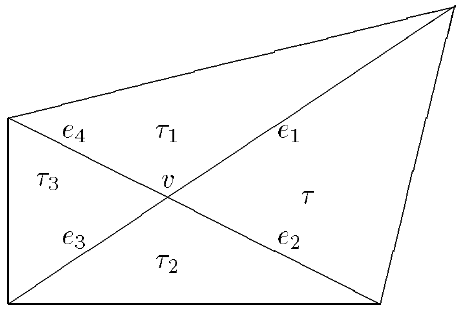


Figure 1. Singular vertex, v.

THEOREM 1. Let $\Pi, \mathcal{T}$ be as above and suppose Π is simply connected. For $n \geq 2$, we have

$$\dim S_n(\Pi, \mathcal{T}) \geq \frac{1}{2}(n+1)(n+2)T - (2n+1)E_0 + 3V_0 + \sigma$$

where T, E_0, V_0 and σ are the numbers of triangles, interior edges, interior vertices, and singular vertices, respectively, in $\mathcal{T}$.

We prove the formula in Theorem 1 by induction on the number of triangles. It is clearly valid for one triangle (see **Case 1** in Table 1), so suppose it is true for $(\Pi, \mathcal{T})$. We must show that it holds true for $(\Pi', \mathcal{T}')$ where $\mathcal{T}' = \mathcal{T} \cup \{\tau\}$ is a triangulation of $\Pi' = \Pi \cup \tau$. Any simply connected Π can be constructed (see Figure 2) by starting with one triangle (**Case 1.**) and adding a new one, τ, where

Case 2. τ shares one edge with Π or

Case 3. τ shares two edges with Π.

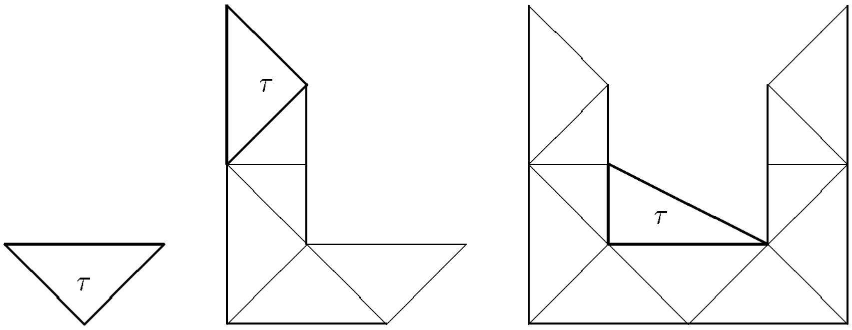


Figure 2. The three cases needed to construct simply–connected domains.

Table 1 summarizes the changes in the quantities in the formula for each case, the last column being the predicted change in dimension that we must verify. The third column of Table 1 should be ignored for the moment, as well as Cases 4–7.

To calculate the change in dimension

$$\dim S_n(\Pi') - \dim S_n(\Pi)$$

(we drop the redundant reference to $\mathcal{T}'$ and $\mathcal{T}$), we study the restriction map

$$\rho : S_n(\Pi') \to S_n(\Pi).$$

We have

$$\dim S_n(\Pi') - \dim S_n(\Pi) = \dim \ker \rho - \operatorname{codim} \operatorname{image} \rho. \tag{2.1}$$

We now use this to verify the formulae given in Table 1. In the following, we use the notation

$$\mathcal{P}_n = \text{space of polynomials of degree} \le n \text{ in two variables}$$

plus the shorthand notation from Table 1: $N = \frac{1}{2}(n+1)(n+2)$ and $K = (2n+1)$. Note that $N = \dim \mathcal{P}_n$.

Case number	increase in no. interior edges	$\sum_{\tau\in\mathcal{T}'} \ell_{\mathcal{T}'}(v)$ $- \sum_{\tau\in\mathcal{T}} \ell_{\mathcal{T}}(v)$	$\dim S_n(\Pi', \mathcal{T}')$ $- \dim S_n(\Pi, \mathcal{T})$
1	0	0	N
2	1	0	$N - K$
3	2	1	$N - 2K + 3 + \sigma$
4	1	-1	$N - K - 3$
5	0	-1	$N - 3$
6	0	-2	$N - 6$
7	0	-3	$N - 9$

Table 1. The change in dimension resulting from adding a triangle, τ. $N = (n+1)(n+2)/2, K = 2n+1, \sigma =$ difference of numbers of singular vertices in $\mathcal{T}'$ and $\mathcal{T}$

Case 1. Obvious for all n.

Case 2. Let τ_1 be the triangle in Π sharing an edge with τ. Given a function ϕ in $S_n(\Pi)$ we can extend it to τ from τ_1 as a polynomial. This proves that the restriction map ρ is onto. Thus, we just have to calculate the dimension of $\ker\rho$. Given ϕ in $\ker\rho$, we know ϕ must vanish to second order on the edge joining τ_1 and τ. Conversely, given any polynomial P that vanishes to second order on that edge, we can construct a $\phi \in \ker\rho$ by setting $\phi = P$ in τ and $\phi = 0$ in Π. Thus $\ker\rho$ is isomorphic to the space of all polynomials of degree n that vanish to second order on a line, which in turn is isomorphic to $\mathcal{P}_{n-2}$. Thus

$$\dim\ker\rho = \dim\mathcal{P}_{n-2} = N - K.$$

This shows the formula is true in Case 2 for $n \geq 2$.

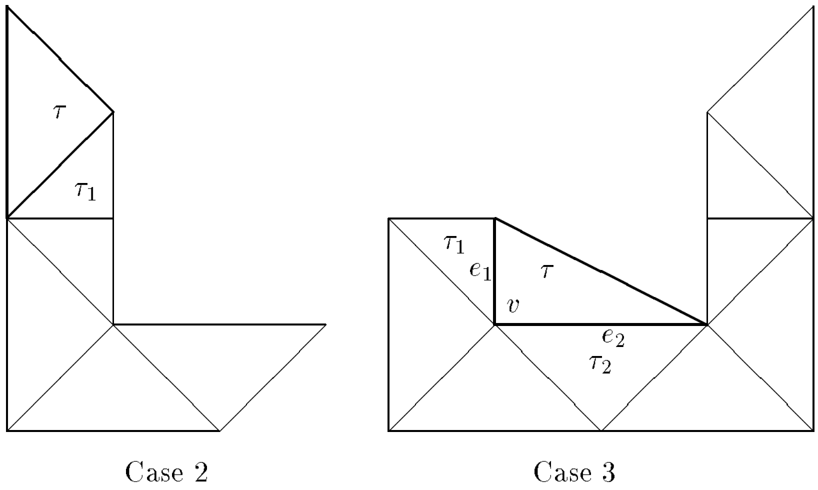


Figure 3. Notation for Cases 2 and 3.

Case 3. In this case, $\ker\rho$ is isomorphic to the space of polynomials vanishing on two lines, so $\ker\rho \simeq \mathcal{P}_{n-4}$ $(n \geq 4)$ and

$$\dim\ker\rho = N - 2K + 4;$$

this formula is valid also for $n = 2$ and 3 since $\ker\rho = \{0\}$ for these n.

Let τ_1 and τ_2 be the two triangles in Π sharing an edge with τ, and let v be the vertex where the three triangles meet. Then we must show that $\operatorname{codim}\operatorname{image}\rho = 0$ or 1 depending on whether v is a singular or nonsingular vertex in $\mathcal{T}'$. This depends on whether or not a function ϕ in $S_n(\Pi)$ can be extended in a C^1 fashion to Π'. Let e_1 (respectively, e_2) be the edge shared by τ and τ_1 (respectively, τ_2). Given ϕ in $S_n(\Pi)$, we must have

$$\partial_{e_2}(\phi\,|_\tau - \phi\,|_{\tau_1}) = 0 \text{ on } e_1 \text{ and } \partial_{e_1}(\phi\,|_\tau - \phi\,|_{\tau_2}) = 0 \text{ on } e_2$$

because ϕ is C^1 (∂_{e_i} denotes differentiation in the direction parallel to e_i). Now differentiating the first equation in the e_1 direction, the second in the e_2 direction, we find

$$\partial_{e_1}\partial_{e_2}(\phi\,|_{\tau_1} - \phi\,|_{\tau_2}) = 0 \text{ at } v \tag{2.2}$$

where v is the vertex at which e_1 and e_2 meet. Thus any function ϕ in $S_n(\Pi)$ must satisfy (2.2) if it is to extend. LEMMA 1. Let $n \geq 2$ and let τ be as in Case 3. A function ϕ in $S_n(\Pi)$ extends to a function in $S_n(\Pi')$ if and only if (2.2) holds.

Proof: "Only if" has just been proved, so we now do the "if." It is sufficient to consider the case when $\Pi = \tau_1 \cup \tau_2$ (and $\Pi' = \Pi \cup \tau$): any $\phi \in S_n(\Pi)$ for a general Π can be viewed as in $S_n(\tau_1 \cup \tau_2)$. If it can be extended to $S_n(\tau_1 \cup \tau_2 \cup \tau)$, this extension on τ will suffice to extend ϕ to $S_n(\Pi')$. Thus we now consider the space $S_n(\tau_1 \cup \tau \cup \tau_2)$. Using Cases 1 and 2, we can calculate that

$$\dim S_n(\tau_1 \cup \tau \cup \tau_2) = 3N - 2K.$$

Now let $\mu : S_n(\tau_1 \cup \tau \cup \tau_2) \to \mathcal{P}_n \times \mathcal{P}_n$ be given by $\mu(\phi) = (\phi\,|_{\tau_1}, \phi\,|_{\tau_2})$. Define $\nu : \mathcal{P}_n \times \mathcal{P}_n \to \mathbf{R}^4$ by

$$\nu(P_1, P_2) = (P_1 - P_2, \partial_{e_1}(P_1 - P_2), \partial_{e_2}(P_1 - P_2), \partial_{e_1}\partial_{e_2}(P_1 - P_2))$$

evaluated at v. We have (by the "only if" part)

$$\operatorname{image} \mu \subset \ker \nu$$

and we want to prove equality. We do this by showing that $\operatorname{image} \mu$ and $\ker \nu$ have the same dimension. For the first, we have

$$\begin{aligned} \dim \operatorname{image} \mu &= \dim S_n(\tau_1 \cup \tau \cup \tau_2) - \dim \ker \mu \\ &= 3N - 2K - \dim \mathcal{P}_{n-4} \\ &= 2N - 4 \end{aligned}$$

where we have used the convention $\mathcal{P}_{-1} = \mathcal{P}_{-2} = \{0\}$. (The fact that $\ker \mu = \mathcal{P}_{n-4}$ follows from the fact that $\ker \mu = \ker \rho$.) To show that $\dim \ker \nu = 2N - 4$, we must simply show that ν is onto. To do this, choose Cartesian coordinates such that e_2 lies on the x-axis and e_1 lies on the line $x = \alpha y$. Let $P(x, y) = a + bx + cy + dxy$. We find

$$\nu(P, 0) = (a, \alpha b + c, b, d).$$

This shows ν is onto and completes the proof of Lemma 1.

Let δ be the linear functional on $S_n(\Pi)$ given by

$$\delta\phi = \partial_{e_1}\partial_{e_2}(\phi\,|_{\tau_1} - \phi\,|_{\tau_2})(v). \tag{2.3}$$

Lemma 1 says that $\ker \delta = \operatorname{image} \rho$, so we have the following corollary to it.

COROLLARY 1. For $n \geq 2$, if τ is added to Π as in Case 3 to form Π', we have

$$\dim S_n(\Pi') - \dim S_n(\Pi) = N - 2K + 4 - \operatorname{codim}\, \ker \delta,$$

where δ is the linear functional on $S_n(\Pi)$ given in (2.3).

If we could show that δ is nontrivial precisely when v is nonsingular in Π', then we would have a proof of equality in (1.3). To complete the proof of Theorem 1, we must only show that δ is identically zero if v is singular. So, assume v is singular, and let τ_3 be the fourth triangle at v (see Figure 1). The edge between τ_1 and τ_3 is co–linear with e_2, so for ϕ in $S_n(\Pi)$ we have

$$\partial_{e_1}(\phi\,|_{\tau_1} - \phi\,|_{\tau_3}) = 0 \text{ on } e_2.$$

Differentiating zero, we find

$$\partial_{e_2}\partial_{e_1}(\phi\,|_{\tau_1} - \phi\,|_{\tau_3})(v) = 0.$$

Similarly, we have

$$\partial_{e_1}\partial_{e_2}(\phi\,|_{\tau_3} - \phi\,|_{\tau_2})(v) = 0,$$

so $\delta\phi = 0$.

3 Regular Mesh Examples

In this section we give a simple application of Corollary 1 to indicate its use. The results contained here have also been obtained by other means, cf. Chui [7] and references therein. Let us construct two triangulations of the plane, $\mathbf{R}^2$. Let $\mathcal{T}_1$ be the triangulation generated by the lines $\{x = k : k \in Z\}$, $\{y = k : k \in Z\}$, and $\{x - y = k : k \in Z\}$. Let $\mathcal{T}_2$ be the triangulation obtained by adding the lines $\{x+y = k : k \in Z\}$. Consider an $r \times r$ square $\Pi_r = \{j_0 \le x \le j_0+r, k_0 \le y \le k_0+r\}$.

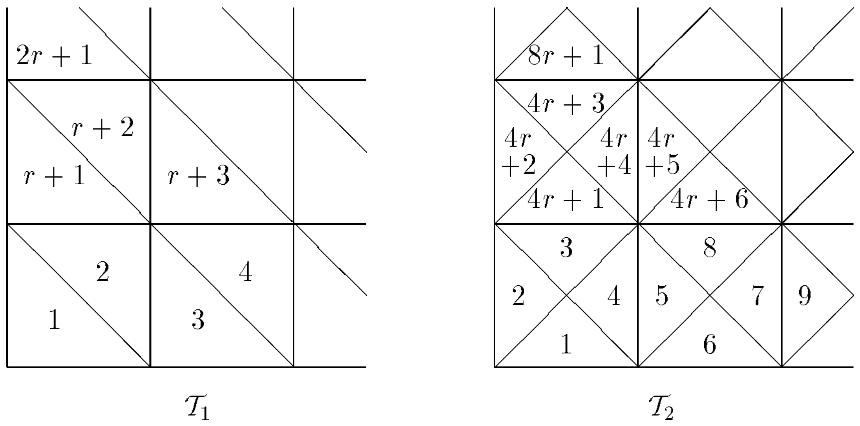


Figure 4. Regular mesh examples.

THEOREM 2. Equality holds in the formula (1.3) for $\dim S_n(\Pi_r, \mathcal{T}_i)$, $i = 1, 2$, for all $n \ge 2$ (r any positive integer).

Proof: We can construct $(\Pi_r, \mathcal{T}_i)$ using only Cases 1–3 as indicated by the ordering of triangles depicted in Figure 4. The induction step was proved for all $n \ge 2$ in Case 1 and Case 2, and Case 3 was proved modulo knowing that the linear functional

δ is nontrivial if the new interior vertex is nonsingular. To see that this is always the case for $\mathcal{T}_1$ and $\mathcal{T}_2$, let us introduce the following.

DEFINITION 2. Let τ be added to Π as in Case 3. We say Case 3 is *open* if one of the triangles in Π sharing an edge with τ (call it τ_1) has a boundary edge in $\Pi \cup \tau$ and the non–boundary edge of τ_1 (if any) is not co–linear with the second edge that τ shares with Π.

LEMMA 2. If τ is added to Π in an open Case 3, then there is a $\phi \in S_2(\Pi)$ such that $\delta\phi \neq 0$ (see 2.3 for the definition of δ).

Proof: Choose coordinates so that the edge (e_1 in Figure 3) where τ and τ_1 meet is on the y-axis and the second edge (e_2 in Figure 3) that τ and Π share lies on the line $y = \alpha x$. Let β and γ be chosen so that the non–boundary edge of τ_1 lies on $y = \beta x + \gamma$ (if there is no non–boundary edge, choose any $\beta \neq \alpha$). Note that $\alpha \neq \beta$ since this edge is assumed not to be co–linear with e_2. Let Q be the quadratic given by $Q(x, y) = (y - \beta x - \gamma)^2$, we define $\phi \in S_2(\Pi)$ by setting $\phi = Q$ in τ_1 and zero elsewhere. Then

$$\delta\phi = (\alpha\partial_{yy} + \partial_{xy})Q(0,0) = 2(\alpha - \beta) \neq 0.$$

This completes the lemma.

The constructions in Figure 5 involve only open or (for $\mathcal{T}_2$) singular Case 3's, so Lemma 2 combined with Corollary 1 in Section 2 completes Theorem 2.

Counting the triangles, edges, etc., in Π_r, we find for $n \geq 2$ (N and K are as in Table 1)

$$\dim S_n(\Pi_r, \mathcal{T}_1) = 2r^2N - (3r^2 - 2r)K + 3(r-1)^2 \tag{3.1}$$

and

$$\dim S_n(\Pi_r, \mathcal{T}_2) = 4r^2N - (6r^2 - 2r)K + 7r^2 - 6r + 3. \tag{3.2}$$

COROLLARY 2. On the triangulation $\mathcal{T}_1$ of $\mathbf{R}^2$, there are no C^1 piecewise quadratics with compact support.

Proof: Suppose there were a C^1 piecewise quadratic ϕ with compact support. Choose an $r \times r$ square Π_r large enough so that the support of $\phi \subset \Pi_r$. By translating ϕ, we can construct d^2 linearly independent, C^1 piecewise quadratics in the square Π_{dr} for any integer d. This implies that $\dim S_2(\Pi_{dr}, \mathcal{T}_1) \geq d^2$, contradicting (3.1) with d large enough.

4 An Example Where Equality Fails in (1.3)

Let us begin with an equilateral triangle Π and place inside it a smaller equilateral triangle such that the inscribed circles for the two triangles are concentric and such that the smaller triangle is rotated 60° from the position in which adjacent sides are parallel. Now draw in six lines, two from each vertex of the smaller triangle to the nearest vertices of the larger triangle, and call this triangulation $\mathcal{T}$ (see Figure 7). No matter what sequence $\Pi_1, ..., \Pi_7 = \Pi$ is chosen to construct Π, the last step will always be to add one of the edges of the boundary edges of Π , and this is a closed (i.e., non–open) Case 3. Further, this need be the only closed Case 3, so by techniques used in proving Theorems 1 and 2 we have

$$6 \leq \dim S_2(\Pi, \mathcal{T}) \leq 7$$

since $T = 7, E_0 = 9, V_0 = 3$, and there are no singular vertices. The purpose of this section is to show that

$$\dim S_2(\Pi, \mathcal{T}) = 7. \tag{4.1}$$

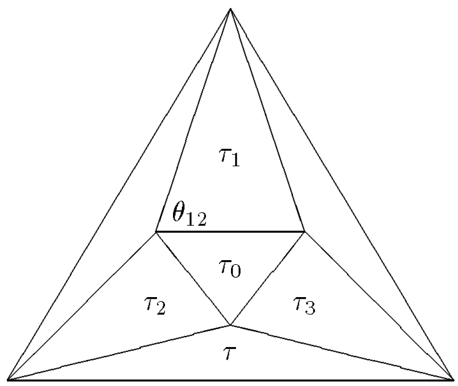


Figure 5. Triangulation where equality fails in (1.3) for S_2.

To prove (4.1), we only have to show that there exists one function ϕ in $S_2(\Pi, \mathcal{T})$ that is not a polynomial. We construct ϕ by setting it zero in the interior equilateral triangle, and in the three triangles sharing an edge with that triangle, we set ϕ equal to (distance from shared edge squared). Now we prove that ϕ can be extended in a C^1 fashion to the remaining three triangles. Since ϕ is symmetric with respect to a rotation by 120°, it suffices to show that it can be extended to one of the remaining triangles, call it τ. Fixing our attention on τ, we choose Cartesian coordinates so that the x-axis is parallel to its boundary edge and so that (0,0) corresponds to its interior vertex, we let P_1 and P_2 be the restriction of ϕ to the triangle to its left and right respectively, and we let $y = -\alpha x, y = \alpha x$ be the non–boundary edges of τ. The key point is that $P_1(x, y) = P_2(-x, y)$, and thus

$$P_1(x, y) - P_2(x, y) = cxy,$$

where c is some constant (recall P_1 and P_2 vanish to second order at (0,0)). We define the extension Q by

$$Q(x, y) = P_1(x, y) - \frac{1}{4}(c/\alpha)(x + \alpha y)^2.$$

Thus Q matches P_1 to second order on the edge $y = -\alpha x$. Further,

$$\begin{aligned} Q - P_2 &= cxy - \frac{1}{4}(c/\alpha)(x + \alpha y)^2 \\ &= -\frac{1}{4}(c/\alpha)(x - \alpha y)^2, \end{aligned}$$

so Q matches P_2 to second order on the other edge, where $y = \alpha x$. Defining $\phi \mid_\tau = Q$, we complete the definition of ϕ. ϕ is clearly not a polynomial, since it is $\equiv 0$ in one triangle and nonzero elsewhere.

Let us analyze this example in more detail to determine whether or not the symmetry is required. So consider a general triangulation similar to the example above (see Figure 5), but without any symmetry. Let τ_0 be the triangle with three interior vertices, let τ_1, τ_2, τ_3 be the triangles with two interior vertices, and let e_i be the edge $\tau_0 \cap \tau_i$. Let τ be the triangle sharing an edge with τ_2 and τ_3.

By Lemma 2, we can calculate the dimension of $S_2(\Pi \setminus \tau)$ to be 7 if at each interior vertex no two edges meeting there are co–linear. (We are interested only in small perturbations of the symmetric triangulation, so we make this assumption.) Let L_i be the linear polynomial that vanishes on e_i and has normal slope 1. The non–polynomial element of $S_2(\Pi \setminus \tau)$ may be represented by the function ϕ that equals 0 in τ_0 and $\alpha_i L_i^2$ in τ_i, with the relationship of the α_i's determined by the geometry of the triangulation. (<u>Proof</u>: If $\psi \in S_2(\Pi \setminus \tau)$, let $P = \psi \mid_{\tau_0}$ and set $\phi = \psi - P$. Then $\phi \equiv 0$ in τ_0, hence must be of the form $\alpha_i L_1^2$ in τ_i since it is C^1.) The relationship between the α_i's is determined by the condition that $\alpha_1 L_1^2$ and $\alpha_2 L_2^2$ (resp. $\alpha_1 L_1^2$ and $\alpha_3 L_3^2$) must be extendible across the triangle between τ_1 and τ_2 (resp. τ_1 and τ_3). We have $\dim S_2(\Pi) = 7$ when this relationship implies that $\alpha_2 L_2^2$ and $\alpha_3 L_3^2$ are extendible across the triangle between τ_2 and τ_3. The dimension of $S_2(\Pi)$ is clearly invariant under a linear transformation in $\mathbf{R}^2$, so we may assume that τ_0 is an equilateral triangle. Let θ_{ij} be the angle formed by e_i and the edge joining τ_i and the triangle between τ_i and τ_j. (Thus θ_{ij} is defined for $j \neq i$, $i = 1, 2, 3$. These six angles determine the triangulation, once we fix τ_0.) Then $\alpha_i L_i^2$ and $\alpha_j L_j^2$ extend between τ_i and τ_j if and only if

$$\alpha_i(\beta_{ij} + \sqrt{3})(\beta_{ji} - \sqrt{3}) = \alpha_j(\beta_{ji} + \sqrt{3})(\beta_{ij} - \sqrt{3}) \tag{4.2}$$

where $\beta_{rs} = \tan(\theta_{rs} - \frac{\pi}{3})$.[1]

It is thus clear that in the symmetric case, i.e., $\theta_{ij} = \theta_{ji}$, there is a nontrivial solution $\alpha_1 = \alpha_2 = \alpha_3 \neq 0$, and $\dim S_2(\Pi) = 7$. This represents a three–parameter family of triangulations, $\mathcal{T}$, having the property that $\dim S_2(\Pi, \mathcal{T}) = 7$. To see that the above matrix can be made invertible by a small perturbation of the symmetric case, let $\gamma = \beta_{12} = \beta_{23} = \beta_{31}$ and let $\gamma + \varepsilon = \beta_{21} = \beta_{32} = \beta_{13}$. If we define $a = \gamma + \sqrt{3}$ and $b = \gamma - \sqrt{3}$, then the equations become

$$\begin{pmatrix} a(b+\varepsilon) & -(a+\varepsilon)b & 0 \\ 0 & a(b+\varepsilon) & -(a+\varepsilon)b \\ (a+\varepsilon)b & 0 & a(b+\varepsilon) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

[1]The condition that no two edges meeting at a vertex are co–linear is equivalent to $\beta_{ij} \neq \sqrt{3}$.

The determinant of this matrix is

$$[a(b+\varepsilon)]^3 - [(a+\varepsilon)b]^3 = (6\sqrt{3}a^2b^2)\varepsilon + \mathcal{O}(\varepsilon^2).$$

We must have $\frac{\pi}{3} < \theta_{12} < \frac{\pi}{2}$, so $a > 0$ and $b < 0$, and the matrix is invertible for sufficiently small $\varepsilon \neq 0$. This implies $\alpha_1 = \alpha_2 = \alpha_3 = 0$, so that $\dim S_2(\Pi, \mathcal{T}_\varepsilon) = 6$. We note that the existence of a perturbation, $\mathcal{T}_\varepsilon$, such that $\dim S_2(\Pi, \mathcal{T}_\varepsilon) = 6$ also follows from [3] and [17].

5 C^1 Piecewise Cubics (n=3)

We now show that for any configuration of $\mathcal{T}$ as in Figure 5, equality holds in the formula (1.3) for the dimension for $S_3(\Pi, \mathcal{T})$, at least under the condition that no two edges meeting at a vertex are co–linear. All we must show is that in the three Case 3 steps required to build Π, the linear functional δ in (2.3) is nontrivial. This is proved by the following.

LEMMA 3. Let τ_1, τ_2, τ_3 meet at a vertex v such that τ_1 and τ_2 (resp. τ_2 and τ_3) share another vertex v_2 (resp. v_3) and such that τ_1 and τ_3 do not share an edge. Let v_1 be the third vertex of τ_1 and let v_4 be the third vertex of τ_3. Then there is a function $\phi \in S_3(\tau_1 \cup \tau_2 \cup \tau_3)$ such that

1) $\phi = 0$ (second order) on the edges $\overline{v_i v_{i+1}}$, $i = 1, 2, 3$, and

2) if e is a direction not co–linear with $\overline{v_1 v_2}$, then $\partial_e \partial_{\overline{v_1 v_2}} \phi(v_1) \neq 0$.

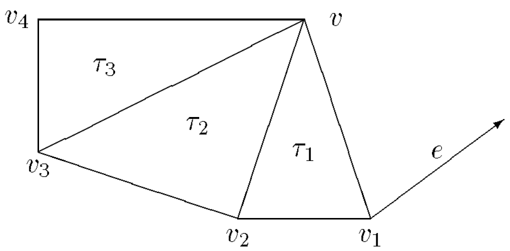


Figure 6: Notation for Lemma 3.

<u>Proof</u>: Choose coordinates so that $v = (0,0)$ and $v_i = (x_i, y_i), i = 1, ..., 4$, with $y_1 = y_2$. In these coordinates, we may write $\partial_e = \alpha \partial_x + \partial_y$ since e is not co–linear with the line $y = y_1$. Let L_i be the linear function that is zero on $\overline{v_i v_{i+1}}$:

$$L_i(x,y) = (y_{i+1} - y_i)(x - x_i) - (x_{i+1} - x_i)(y - y_i).$$

In each τ_i, we want ϕ to be of the form

$$P_i(x,y) = L_i(x,y)^2(a + bx + cy). \tag{5.1}$$

We claim that, for any point (x_0, y_0) on $\overline{v_i v_{i+1}}$, such a polynomial satisfies

$$P_i(x,y) = \frac{L_i(x,y)^2}{L_i(0,0)^2}([3 + x_0\partial_x + y_0\partial_y]P_i)(0,0) + H_3(x - x_0, y - y_0), \tag{5.2}$$

where H_3 is homogeneous of order 3. Since $\phi\mid_{\tau_1}= P_1$, we can compute

$$\begin{aligned}\partial_e \partial_{\overline{v_1 v_2}} \phi(v_1) &= (\alpha\partial_x + \partial_y)(x_1\partial_x + y_1\partial_y)P_1(x_1, y_1) \\ &= y_1 \frac{2(x_2 - x_1)^2}{(x_2 y_1 - x_1 y_2)^2}[3 + x_1\partial_x + y_1\partial_y]P_1(0,0).\end{aligned}$$

Given P_1, we know by Lemma 1 that we can find a cubic P_2 that matches P_1 to second order on $\overline{v_2 v}$ and zero to second order on $\overline{v_2 v_3}$ if

$$[3 + x_2\partial_x + y_2\partial_y]P_1(0,0) = 0,$$

in view of (5.2) applied with $(x_0, y_0) = (x_2, y_2)$. We may assume that v_3 is not on the line $\overline{v_1 v_2}$, since if it were, we could simply let $P_2 = P_1$, effectively reducing the number of triangles. Since P_2 is necessarily of the form (5.1), we can now find a cubic P_3 matching P_2 to second order on $\overline{v_3 v}$ and zero to second order on $\overline{v_3 v_4}$ provided

$$[3 + x_3\partial_x + y_3\partial_y]P_2(0,0) = 0.$$

All the $P_i' s$ agree to second order at $(0,0)$, so to complete the lemma, we must find a solution of

$$\begin{pmatrix} 3 & x_1 & y_1 \\ 3 & x_2 & y_2 \\ 3 & x_3 & y_3 \end{pmatrix} \begin{pmatrix} \phi(0,0) \\ \partial_x\phi(0,0) \\ \partial_y\phi(0,0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

This matrix is invertible precisely when the $(x_i, y_i)' s$ are not co–linear, as we have assumed. Now we prove the representation (5.2). We write

$$\begin{aligned}P_i(x,y) &= L_i(x,y)^2(a + bx + cy) \\ &= L_i(x,y)^2(a + bx_0 + cy_0) + H_3(x - x_0, y - y_0),\end{aligned}$$

where $H_3(x - x_0, y - y_0) = L_i(x,y)^2(b(x - x_0) + c(y - y_0))$. We have

$$a = P_i(0,0)/L_i(0,0)^2;$$

differentiating, we find

$$\begin{aligned}\partial_x P_i(0,0) &= a\partial_x L_i^2(0,0) + bL_i(0,0)^2 = 2aL_i(0,0)(y_{i+1} - y_i) + bL_i(0,0)^2 \\ &= 2P_i(0,0)\frac{y_{i+1} - y_i}{L_i(0,0)} + bL_i(0,0)^2 \quad \text{and} \\ \partial_y P_i(0,0) &= a\partial_y L_i^2(0,0) + cL_i(0,0)^2 = 2aL_i(0,0)(x_{i+1} - x_i) + cL_i(0,0)^2 \\ &= 2P_i(0,0)\frac{x_{i+1} - x_i}{L_i(0,0)} + cL_i(0,0)^2.\end{aligned}$$

Adding, we have

$$\begin{aligned}L_i(0,0)^2(a + x_0 b + y_0 c) &= [3 + x_0\partial_x + y_0\partial_y]P_i(0,0) \\ &\quad - 2P_i(0,0)[1 + \frac{x_0(y_{i+1} - y_i) - y_0(x_{i+1} - x_i)}{L_i(0,0)}].\end{aligned}$$

The term in brackets is equal to

$$1 + \frac{L_i(x_0, y_0) - L_i(0,0)}{L_i(0,0)} = L_i(x_0, y_0)/L_i(0,0).$$

Since (x_0, y_0) was assumed to lie on the line $L_i(x, y) = 0$, this completes the proof of (5.2), and the lemma is complete.

DEFINITION 3. Let τ be added to $\Pi, \mathcal{T}$ as in Case 3, and let τ_1 be one of the triangles sharing an edge e_1 with τ. Let e_2 be the other edge of τ shared with Π, let e_3 be the other edge of τ_1 that meets e_1 at the common vertex of e_1 and e_2, and let v be the other vertex of e_1. Then we say the Case 3 is *unconstrained* if e_2 and e_3 are not co–linear and the number of triangles in $\mathcal{T}$ meeting at v is 3 or less (i.e., excluding τ).

THEOREM 3. Suppose $(\Pi, \mathcal{T})$ can be constructed via Cases 1–3 using only unconstrained Case 3's. Then the formula given in Theorem 1 gives the dimension of $S_3(\Pi, \mathcal{T})$.

<u>Proof</u>: Apply Lemma 3 and Corollary 1.

<u>Remark</u>: Carl de Boor [5] has shown that Lemma 3 can be proved using Bernstein-Bezier coordinate representation, cf. [4].

In the cubic case, the techniques for dimension counting can be extended to include nonsimply–connected domains. To do so, we must generalize to the case when $\mathcal{T}$ is an arbitrary rectilinear sub-complex of $\mathbf{R}^2$. With $\Pi = \cup\mathcal{T}$, we again define $S_n(\Pi, \mathcal{T})$ as the space of C^1 piecewise polynomials of degree n, but we must say what it means to be C^1 at the self-intersections of $\partial\Pi$. At such points, we require that the restrictions of a $\phi \in S_n$ to the various abutting triangles agree to second order. We will calculate $\dim S_n(\Pi, \mathcal{T})$ in this setting. To do so, we recall the notion of the <u>link</u> of a vertex. Given a vertex v in $\mathcal{T}$, let $\tau_1, ..., \tau_r$ be the triangles in $\mathcal{T}$ that meet at v, and let e_j be the edge of τ_j not containing v. The <u>link</u> of v with respect to $\mathcal{T}$ is defined to be $e_1 \cup \ldots \cup e_r$, and the <u>order</u> of the link is given by

$$\ell_{\mathcal{T}}(v) = 1 - \chi(e_1 \cup \ldots \cup e_r), \tag{5.3}$$

where χ is the Euler characteristic. Definition 1 makes sense for the general $\mathcal{T}$ we are now considering (note that (i) in Definition 1 is equivalent to $\ell_{\mathcal{T}}(v) = 1$).

THEOREM 4. Let $\mathcal{T}$ be a rectilinear sub-complex of $\mathbf{R}^2$ and let $\Pi = \cup\mathcal{T}$. For $n \geq 3$, we have

$$\dim S_n(\Pi, \mathcal{T}) \geq \frac{1}{2}(n+1)(n+2)T - (2n+1)E_0 + 3\sum_{v \in \mathcal{T}} \ell_{\mathcal{T}}(v) + \sigma$$

where T, E_0, and σ are as in Theorem 1 and $\ell_{\mathcal{T}}(v)$ is defined by (5.3). (The sum is over all vertices in $\mathcal{T}$.)

When Π is a domain with non–intersecting polygonal boundary and $\mathcal{T}$ is a triangulation of Π, then $\ell_{\mathcal{T}}(v) = 0$ or 1 depending on whether v is a boundary or interior vertex of $\mathcal{T}$. Thus Theorem 4 implies Theorem 1 holds (for $n \geq 3$) for nonsimply–connected domains. We prove the formula in Theorem 4 again by induction on the number of triangles as we did Theorem 1. We must show that it holds true for $(\Pi', \mathcal{T}')$ where $\mathcal{T}' = \mathcal{T} \cup \tau$ is also a rectilinear sub-complex of $\mathbf{R}^2$ and $\Pi' = \Pi \cup \tau$. Case 1 should be modified to allow the possibility that τ is simply disjoint from Π, but the verification is again trivial. Case 2 is unmodified, but in Case 3 we should consider the possibility that v is not an interior point in Π', which is equivalent to $\ell_{\mathcal{T}'}(v) \leq 0$. However, an examination of the derivation of (2.2) shows it to be valid in this case as well. Thus Case 3 is as before. To the list of the possible ways in which τ can join Π, we must add the following (see Figure 7) to those studied in the proof of Theorem 1:

Case 4. τ shares one edge and one vertex with Π, or

Cases 5, 6 & 7. τ shares one, two, or three vertices, respectively with Π.

Table 1 summarizes the changes in the quantities in the formula for each case, the last column being the predicted change in dimension that we must verify. We refer to the proof of Theorem 1 for notation.

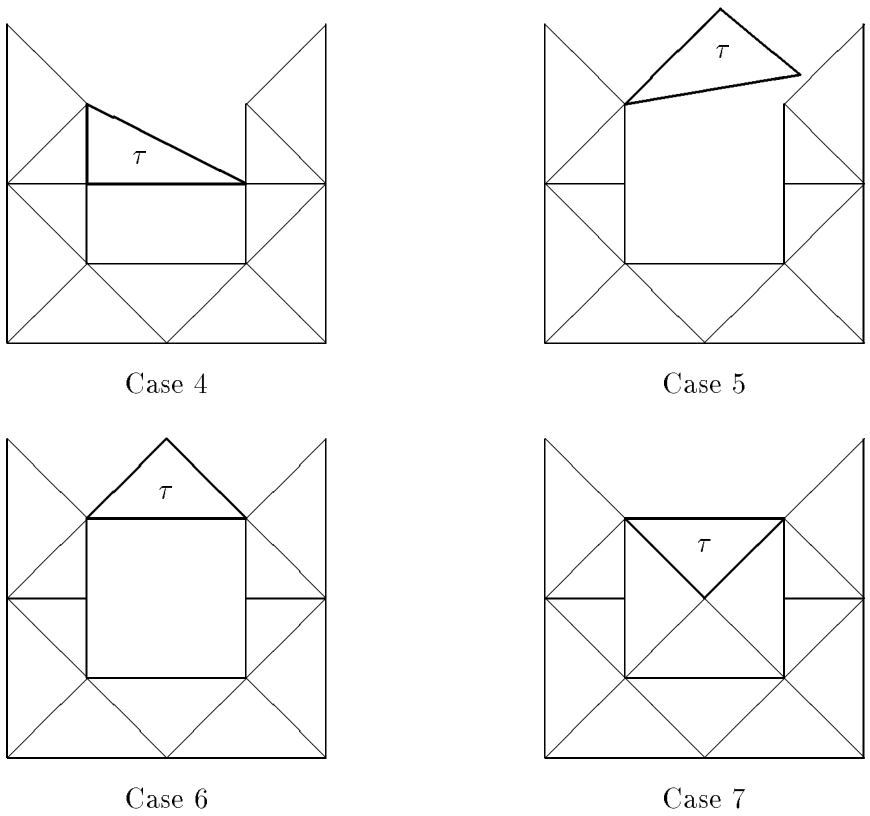


Figure 7. The four remaining cases needed to construct general domains.

Case 4. For $n \geq 3$, we can extend any function ϕ in $S_n(\Pi)$ to Π'. To see this, let the edge shared by τ and Π be the y-axis, with the other vertex of τ shared with Π being the point $(x_0, 0)$. Let τ_1 be the triangle in Π sharing an edge with τ. Let $P = \phi \mid_\tau$, and define

$$Q(x, y) = P(x, y) + x^2(a + bx + cy),$$

where a, b, c are chosen so that $Q - \phi$ is zero to second order at $(x_0, 0)$. Q is the required extension. To see that we can choose Q to match ϕ at $(x_0, 0)$, we simply have to check that the cubic $C(x, y) = x^2(a + bx + cy)$ can have arbitrary value and gradient at $(x_0, 0)$. Computing, we have

$$\begin{aligned} C(x_0, 0) &= x_0^2(a + x_0 b) \\ C_x(x_0, 0) &= x_0(2a + 3x_0 b) \\ C_y(x_0, 0) &= x_0^2 c. \end{aligned}$$

These are clearly independent. Thus, the restriction map is onto. Further, $\ker \rho$ consists of polynomials that vanish to second order on $x = 0$ and at $(x_0, 0)$, and (as the cubic C above shows) it has dimension $N - K - 3$ if $n \geq 3$. This proves the formula in Case 4 for $n \geq 3$.

Cases 5, 6 & 7. Given any three points in the plane that are not co–linear, we can find a cubic polynomial with arbitrary value and gradient at each point. In fact, we can specify the value and gradient at the three points and, for example, the value at any point not on the three lines formed by the three points. This is an example of the well known (cf. [8,16,18]) generalization of Hermite interpolation to two dimensions. Therefore the restriction map ρ is onto for $n \geq 3$. Further, the kernel of ρ clearly has dimension $N - 3\nu$, where ν is the number of vertices shared by τ and Π. (To see this, view $\ker \rho$ as parameterized by the space of polynomials, $\phi \mid_\tau$ for $\phi \in S_n(\Pi')$, vanishing to second order at ν points, not co–linear if $\nu = 3$. This is also the kernel of the map that evaluates a polynomial and its gradient at the ν points. This map is onto, for $n \geq 3$, by our argument above. Hence, the dimension of its kernel is the difference of the dimensions of the domain and range.) This verifies the dimension formula for Cases 5–7 if $n \geq 3$ and completes the proof of Theorem 4.

Combining the technique of proof above with Lemma 3 yields the following:

THEOREM 3′. Suppose $(\Pi, \mathcal{T})$ can be constructed via Cases 1–7 using only unconstrained Case 3's. Then equality holds in the formula (1.3) for the dimension of $S_3(\Pi, \mathcal{T})$.

6 The Clough–Tocher and Fraeijs de Veubeke–Sander Macro–Cubics

Let Π_1 be a triangle, and let $\mathcal{T}_1$ be the triangulation obtained by joining an interior point of Π_1 to each vertex with an edge. We shall prove that, given any 12 numbers

a^i_j, b_j $(i,j = 1,2,3)$ there is a unique $\phi \in S_3(\Pi_1, \mathcal{T}_1)$ such that

$$\phi(v_j) = a^1_j, \quad \partial_x\phi(v_j) = a^2_j, \quad \partial_y\phi(v_j) = a^3_j, \quad \& \quad \partial_{n_j}\phi(m_j) = b_j, \tag{6.1}$$

where $\{v_j\}$ are the vertices of Π_1, $\{m_j\}$ are the midpoints of the edges of Π_1, and n_j is the normal to the j-th edge. The triangle Π_1, together with the associated space $S_3(\Pi_1, \mathcal{T}_1)$ and the representation (6.1) is called the Clough–Tocher macro–cubic finite element [8,18].

Let Π_2 be any convex quadrilateral, and let $\mathcal{T}_2$ be the triangulation obtained by drawing in the diagonals. We shall prove that, given any 16 numbers a^i_j, b_j $(i = 1,2,3, j = 1,...,4)$, there is a unique $\phi \in S_3(\Pi_2, \mathcal{T}_2)$ such that (6.1) holds for $j = 1,...,4$, where v_j, etc., are the corresponding quantities for Π_2. This aggregate is called the Fraeijs de Veubeke–Sander macro–cubic finite element [8,18].

Both of these parametrizations of $S_3(\Pi_k, \mathcal{T}_k)$ have been known for a long time; for references to proofs in the mathematical literature see [8]. Here we give a unified proof using the results and techniques of the previous section. By Theorem 4, we conclude that the number of parameters in (6.1) is equal to the dimension of the space of C^1 piecewise cubics to be represented. So to show that the "finite elements" are well defined (cf. [8]) it suffices to show that any function ϕ in $S_3(\Pi_k, \mathcal{T}_k)$ with all the parameters a^i_j, b_j equal to zero must be identically zero. To see this, let us choose coordinates so that the interior vertex of $\mathcal{T}_k$ is (0,0). Using the representation (5.2) of the restriction of ϕ to each triangle in $\mathcal{T}_k$, we find that for ϕ to be C^1 we must have

$$(3 + x_i\partial_x + y_i\partial_y)\phi(0,0) = 0, \quad i = 1,2,3,$$

where $(x_i, y_i) = v_i$. Since v_1, v_2, v_3 span a triangle, ϕ is zero to second order at (0,0), and hence it is $\equiv 0$ (see the proof for Case 4 in the previous section).

7 A nodal basis for $S_4(\Pi, \mathcal{T})$

As before, let V_0 and E_0 denote, respectively, the numbers of interior vertices and edges in a planar triangulation $\mathcal{T}$. Further, let V, E and T denote the numbers of vertices, edges, and triangles, respectively, in $\mathcal{T}$. Suppose that $\mathcal{T}$ is a triangulation of a bounded domain $\Pi \subset \mathbf{R}^2$ such that $\partial\Pi$ consists of a finite number of non–intersecting polygonal arcs. Then, obviously, $V - V_0 = E - E_0$. Furthermore, $E + E_0 = 3T$ (for a proof, see 7.6 below). Using these relations, the formula (1.3) for the dimension of $S_4(\mathcal{T})$ becomes

$$\dim S_4(\Pi, \mathcal{T}) \geq 4V + E - V_0 + \sigma. \tag{7.1}$$

This formula suggests $\varphi \in S_4$ could be parametrized by the following nodal values, namely,

(i) the value and x and y derivatives of φ at each vertex in $\mathcal{T}$

(ii) a second (cross) derivative of φ at each vertex

(iii) the value of φ at the midpoint of each edge, except for "omitted" edges, where we omit one edge emanating from each nonsingular interior vertex.

More specifically, (ii) means the following. Let τ be a triangle in $\mathcal{T}$ with a vertex v, and let e_1 and e_2 be the edges of τ meeting at v. Then the "τ-cross derivative of $\varphi \in S_n$ at v" is defined to be

$$\partial_{e_1}\partial_{e_2}(\varphi\,|_\tau)(v). \tag{7.2}$$

In (ii) above, it is meant that, for each vertex v in $\mathcal{T}$, a triangle τ is assigned (having v as a vertex) and the τ-cross derivative at v is used as a nodal variable.

In order to show that the above nodal variables determine a basis for S_4, there is one technicality that we must avoid (see Figure 8).

DEFINITION 4. Let τ be a triangle in $\mathcal{T}$, with edges e_1, e_2 meeting at the vertex v. Let τ_i (i = 1,2) be the other triangles in $\mathcal{T}$ with e_i as an edge, and let e_{i+2} be the other edges of τ_i with v as a vertex. If both pairs e_1, e_4 and e_2, e_3 are co–linear and v is not singular in $\mathcal{T}$, then we say τ is *quasi–singular* in $\mathcal{T}$ at v. If either τ_i does not exist, then τ is not quasi–singular at v.

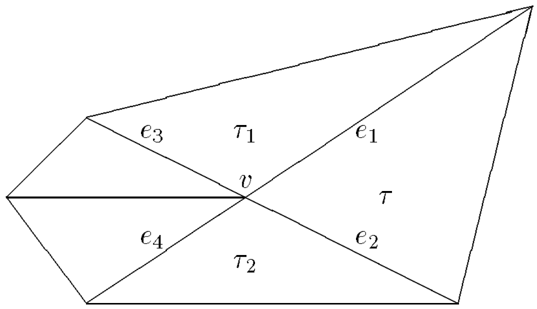


Figure 8. Quasi-singular vertex, v.

THEOREM 5. Let Π and $\mathcal{T}$ be as described above and suppose that $\mathcal{T}$ has no quasi–singular vertices. Then there is an "edge omission" as in (iii) above such that, for any "vertex–triangle assignment" in (ii), the nodal variables (i–iii) parametrize $S_4(\Pi, \mathcal{T})$. That is, if we number the nodal variables $1, 2, ..., E + 4V - V_0 + \sigma$, then there is a basis $\{\varphi_j\}$ of $S_4(\Pi, \mathcal{T})$ such that the j-th nodal value of φ_j is equal to one and the rest are zero.

COROLLARY 3. Under the conditions above on Π and $\mathcal{T}$ in Theorem 5, equality holds in the formulae (1.3) and (7.1) for the dimension of $S_4(\Pi, \mathcal{T})$.

<u>Remark</u>: The proof of Theorem 5 is inductive, and the omissions in (iii) are not specified initially. Thus we have little information about the basis $\{\varphi_j\}$. In principle, one can follow the proof to construct the basis.

<u>Proof</u>: Following the examples in the previous sections, the technique will be inductive. In order to treat nonsimply–connected Π, it is necessary to generalize the result to the case when $\mathcal{T}$ is an arbitrary finite, rectilinear subcomplex of $\mathbf{R}^2$, as was done in Theorem 4 in section 5. In this case, $S_n(\mathcal{T})$ is defined to be the set of functions with a C^1 extension to a neighborhood of $\Pi \equiv \cup\{\tau \in \mathcal{T}\}$ such that, for all

$\tau \in \mathcal{T}$ and $\varphi \in S_n(\mathcal{T})$, $\varphi \mid \tau$ is a polynomial of degree n or less. This simply means that, when two triangles $\tau_i \in \mathcal{T}$, $i = 1, 2$, meet at a common vertex v but do not share a common edge, then each $\varphi \in S_n(\mathcal{T})$ must be such that

$$\varphi \mid_{\tau_1} - \varphi \mid_{\tau_2}$$

vanishes to second order at v. Otherwise, the C^1 continuity condition has the same implications as before.

The nodal variables for $S_4(\mathcal{T})$ in this new context are more complicated and require the use of the following notions, which we repeat from section 5. Let v be any vertex of $\mathcal{T}$and let $\tau_1, ..., \tau_r \in \mathcal{T}$ denote the triangles having v as a vertex, and let e_j be the edge of τ_j such that v is <u>not</u> contained in e_j. The *link* of v with respect to $\mathcal{T}$ is defined to be $e_1 \cup \ldots \cup e_r$ and the *order* of the link is given by

$$\ell(v) = \ell_{\mathcal{T}}(v) = 1 - \chi(e_1 \cup \ldots \cup e_r), \tag{7.3}$$

where χ is the Euler characteristic. Thus $\ell_{\mathcal{T}}(v) = 1$ for an interior vertex, and $\ell_{\mathcal{T}}(v) = 0$ for a boundary vertex of $\mathcal{T}$ if the latter is as in Theorem 5. (In fact, this gives an alternative characterization of the domain regularity assumed there: $\ell_{\mathcal{T}}(v) \geq 0$ for all vertices in $\mathcal{T}$.) If the link of v is disconnected, as can happen for the more general $\mathcal{T}$ now being considered, $\ell_{\mathcal{T}}(v)$ is simply one minus the number of components. With the notation above, let the *star* of v (with respect to $\mathcal{T}$) be given by

$$\mathrm{star}(v) = \mathrm{star}_{\mathcal{T}}(v) = \tau_1 \cup \cdots \cup \tau_r. \tag{7.4}$$

With the above notation, we now give a description of the nodal variables for $S_4(\mathcal{T})$ in the case of a general finite, rectilinear subcomplex $\mathcal{T}$ of $\mathbf{R}^2$. The nodes are identical to (i–iii) except at vertices v such that $\ell(v) < 0$. First, (iii) is interpreted to mean that an edge node is not omitted as long as $\ell(v) \leq 0$ (or if $\ell(v) = 1$ and v is singular). Thus, if e is an edge having vertices v_1 and v_2 such that $\ell_{\mathcal{T}}(v_i) \leq 0, i = 1, 2$, then e is not omitted. The nodes (ii) are actually augmented if $\ell(v) < 0$. In this case, choose a triangle τ_i from each component of $\mathrm{star}(v)$, $i = 1, \ldots, 1 - \ell(v)$. Then the nodal variables include each of the τ_i cross derivatives at v. We now prove that these nodal variables parametrize $S_4(\mathcal{T})$. Theorem 4 gives a lower bound for the dimension of $S_4(\mathcal{T})$:

$$\dim S_4(\mathcal{T}) \geq f_4(\mathcal{T}) := 15T - 9E_0 + 3 \sum_{v \in \mathcal{T}} \ell_{\mathcal{T}}(v) + \sigma \tag{7.5}$$

In order to see that the modified nodal variables (i–iii) described above are precisely $f_4(\mathcal{T})$ in number, we must make a calculation similar the one yielding (7.1). For any finite subcomplex $\mathcal{T}$, think of assigning marbles from each edge to each triangle that it bounds. This takes $2E_0 + (E - E_0)$ marbles (an interior edge is in two triangles, the rest in only one). Furthermore, since each triangle has three edges, each triangle gets three marbles. Thus,

$$3T = 2E_0 + (E - E_0) = E + E_0 \tag{7.6}$$

as before. To get an expression for $E - E_0$, we reason as follows. For each vertex, $v \in \mathcal{T}$, there are exactly $2(1 - \ell_{\mathcal{T}}(v))$ non–interior edges meeting at v. Assigning marbles from vertices to non–interior edges thus requires $2\sum_{\tau\in\mathcal{T}}(1-\ell_{\mathcal{T}}(v))$ marbles. Since each non–interior edge has two vertices, it gets two marbles. Dividing by two thus gives

$$E - E_0 = \sum_{\tau\in\mathcal{T}}(1-\ell_{\mathcal{T}}(v)). \tag{7.7}$$

(In the case that $\partial\Pi$ consists of non–intersecting arcs, then $\ell_{\mathcal{T}}(v)\in\{0,1\}$ for all $v\in\mathcal{T}$, and we find $E-E_0=V-V_0$. However, this can be derived more simply in that case.) Using relations (7.6) and (7.7) in (7.5), we find

$$\begin{aligned} f_4(\mathcal{T}) &= 15T - 9E_0 + 3\sum_{v\in\mathcal{T}}\ell_{\mathcal{T}}(v)+\sigma \\ &= E + 4(E-E_0) + 3\sum_{v\in\mathcal{T}}\ell_{\mathcal{T}}(v)+\sigma \\ &= E + 4V - \sum_{v\in\mathcal{T}}\ell_{\mathcal{T}}(v)+\sigma \end{aligned}$$

This is precisely the number of prescribed nodal variables. To show that these nodal variables parametrize $S_4(\mathcal{T})$, it suffices to show they are *unisolvent* [8], i.e., that if $\varphi\in S_4(\mathcal{T})$ has all nodal variables zero, then $\varphi\equiv 0$. To see why, think of the nodal variables as defining a linear mapping

$$S_4(\mathcal{T})\to\mathbf{R}^{f_4(\mathcal{T})},$$

and unisolvence means that it is injective. Since dim $S_4\geq f_4$, this mapping must be an isomorphism. Therefore the nodal variables form a basis for the dual space of S_4; $\{\varphi_i\}$ is the corresponding dual basis.

As indicated previously, the proof proceeds inductively. If $\mathcal{T}$ consists of one triangle, then it is easy to see that the stated nodal variables determine a polynomial of degree 4: the vanishing of the nodal values (i) and (iii) of $\varphi\in S_4(\mathcal{T})$ implies that φ vanishes on the edges e_i, $i=1,2,3$, of τ ($\mathcal{T}=\{\tau\}$). Thus $\varphi=\Pi_{i=0}^{3}L_i$, where L_0 is an arbitrary linear function and L_i, $i\geq 1$, is a nontrivial linear function vanishing on e_i. Let v_i be the vertex opposite e_i, and interpret indices i modulo 3. Then

$$\partial_{e_i}\partial_{e_{i+1}}\varphi(v_{i+2}) = (\partial_{e_i}L_{i+1})(\partial_{e_{i+1}}L_i)[L_{i+2}L_0](v_{i+2}) \tag{7.8}$$

because $\partial_{e_i}L_i\equiv 0$ and $L_i(v_{i+2})=L_{i+1}(v_{i+2})=0$. Since $\partial_{e_i}L_j\neq 0$ unless $i=j$ and $L_i(v_i)\neq 0$, for all i and j, then the vanishing of the nodal values of type (ii) of φ implies that L_0 vanishes at the vertices of τ. Hence $L_0\equiv 0$ and $\varphi\equiv 0$, and the theorem is proved for one triangle.

Now suppose the theorem has been proved for all rectilinear subcomplexes $\mathcal{T}$ of $\mathbf{R}^2$ consisting of k or fewer triangles, $k\geq 1$. Consider such a complex $\mathcal{T}'$ consisting of $k+1$ triangles. To prove that $S_4(\mathcal{T}')$ has a basis as claimed, we remove a triangle τ from $\mathcal{T}'$, use the fact that $\mathcal{T}=\mathcal{T}'\setminus\{\tau\}$ has a basis of the form (i–iii), and consider the effect of adding τ back to $\mathcal{T}$ to reform $\mathcal{T}'$. The proof breaks down into separate cases, depending on how τ may join $\mathcal{T}$ to form $\mathcal{T}'$. Since $\Pi'=\cup_{\tau\in\mathcal{T}'}\tau$ always has a boundary edge, some τ may be removed from $\mathcal{T}'$ such that τ meets $\mathcal{T}$ in one of the following ways.

Case 1: τ is disjoint from $\mathcal{T}$.
Cases 5–7: τ shares one, two, or three vertices, respectively, with $\mathcal{T}$ (but no edges are shared).
Case 2: τ shares one edge with $\mathcal{T}$.
Case 4: τ shares one edge and one vertex with $\mathcal{T}$.
Case 3: τ shares two edges with $\mathcal{T}$.

Note that we have numbered the cases to be considered in the same way as in previous sections, but they have been listed, and will be studied, in the order of difficulty. If $\mathcal{T}'$ were not a planar triangulation, one would also be forced to consider the case when τ shares three edges with $\mathcal{T}$, but this case is beyond the scope of our current techniques in any case. We now give a case by case verification of the induction step. In the following let

$$\Pi = \cup\{\tau \mid \tau \in \mathcal{T}\} \quad \text{and} \quad \Pi' = \cup\{\tau \mid \tau \in \mathcal{T}'\}.$$

What is to be shown is that, if $\varphi \in S_4(\mathcal{T}')$ has a vanishing set of nodal values (i–iii) for $\mathcal{T}'$ then $\varphi \equiv 0$. That is, we must give a prescription for edge omissions in (iii) for $\mathcal{T}'$ such that, for any choice of cross derivatives in (ii), the set of nodal variables determines $S_4(\mathcal{T}')$. The edge omission for $\mathcal{T}'$ will be derived from the inductively assumed one for $\mathcal{T}$ in the obvious way: We take the same edge omissions as for $\mathcal{T}$ and add any necessary in forming $\mathcal{T}'$ (the latter is required only in Case 3). This edge–omission strategy will be assumed implicitly from now on. Note that the induction hypothesis allows us to assume that the nodal variables of type (ii) for $\mathcal{T}$ correspond to any choice of triangles that we like.

<u>Cases 1, 5, 6 & 7</u>: τ shares i vertices (and no edges) with $\mathcal{T}$, $i = 0, 1, 2$, or 3.

The nodal variables for $\mathcal{T}'$ include those for $\mathcal{T}$ in these cases, and moreover the nodal variables within τ are the same as for a single triangle. Thus the vanishing of the nodal values of $\varphi \in S_4(\mathcal{T}')$ implies that $\varphi|_\Pi \equiv 0$ (by the induction hypothesis) and $\varphi|_\tau \equiv 0$ (by the argument above for one triangle used to start the induction). Therefore $\varphi \equiv 0$ everywhere as claimed.

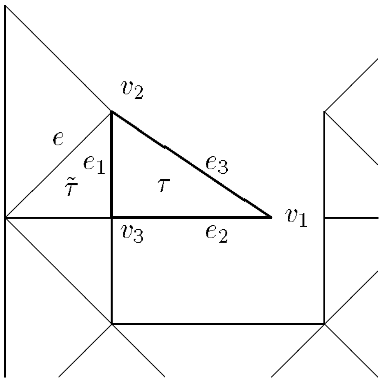
<u>Cases 2 & 4</u>: τ shares one edge (and possibly an additional vertex) with $\mathcal{T}$.

Let e_1 be the shared edge; let e_2 and e_3 be the other edges of τ; and let v_i be the vertex of τ opposite e_i. If neither of the τ-cross derivatives at v_2 or v_3 are nodal variables for $\mathcal{T}'$, then the nodal variables for $\mathcal{T}'$ contain a set of nodal variables for $\mathcal{T}$. In this case, if $\varphi \in S_4(\mathcal{T}')$ has vanishing nodal values, then $\varphi|_\Pi \equiv 0$ by the induction hypothesis. Since φ is C^1, this means that φ vanishes to second order on e_1. Therefore, if we temporarily view τ as a separate entity with its own (fifteen) nodal variables, we see that all of these vanish for φ: The ones located on e_1 vanish since φ vanishes to second order there, and the remaining ones are nodal variables for $\mathcal{T}'$. Therefore $\varphi \equiv 0$ in τ as well, and the proof is complete in this case. Note that it is irrelevant whether $v_1 \in \mathcal{T}$ (Case 4) or not (Case 2).

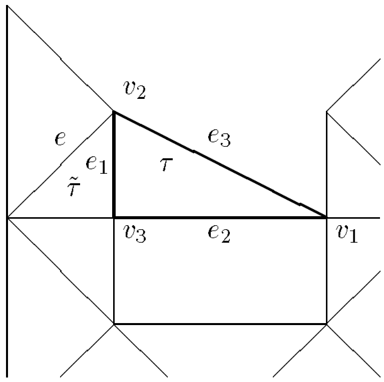
If either of the τ-cross derivatives at v_2 or v_3 is a nodal variable for $\mathcal{T}'$, then the set of nodal variables does not contain a valid set of nodal variables for $\mathcal{T}$. This is because there is exactly one type (ii) nodal variable for $\mathcal{T}'$ at v_2 and v_3, and the τ-cross derivatives are not valid nodal variables for $\mathcal{T}$ (since $\tau \notin \mathcal{T}$). Let $\tilde{\tau}$ be the triangle in $\mathcal{T}$ having e_1 as an edge. For the moment, suppose that the τ-cross derivative at v_2 is a nodal variable for $\mathcal{T}'$. Let e denote the edge of $\tilde{\tau}$ that intersects e_1 at v_2. If the $\mathcal{T}'$ nodal values of $\varphi \in S_4(\mathcal{T}')$ vanish, then φ vanishes on e_1 since it vanishes to second order at v_2 and v_3 and to first order at the midpoint of e_1 (note that neither v_2 or v_3 are interior vertices, so the midpoint is a nodal variable). If we write $\partial_e = \alpha\partial_{e_1} + \beta\partial_{e_3}$, then

$$\begin{aligned}(\partial_e\partial_{e_1}\varphi\,|_{\tilde{\tau}})(v_2) &= (\alpha\partial_{e_1}^2\varphi\,|_{\tilde{\tau}})(v_2) + (\beta\partial_{e_1}\partial_{e_3}\varphi\,|_{\tilde{\tau}})(v_2)\\ &= (\beta\partial_{e_1}\partial_{e_3}\varphi\,|_{\tilde{\tau}})(v_2) = (\beta\partial_{e_1}\partial_{e_3}\varphi\,|_{\tau})(v_2).\end{aligned}$$

Therefore, the vanishing of the τ-cross derivative of φ at v_2 implies the vanishing of the $\tilde{\tau}$-cross derivative of φ at v_2 as well, and the latter is a valid nodal variable for $\mathcal{T}$. Making a similar argument at v_3 (if necessary), we see that the vanishing of the $\mathcal{T}'$ nodal variables of φ implies the vanishing of a corresponding set of $\mathcal{T}$ nodal variables. Thus $\varphi\,|_{\Pi} \equiv 0$, and the remainder of the proof proceeds as before.



Notation for Case 2.



Notation for Case 4.

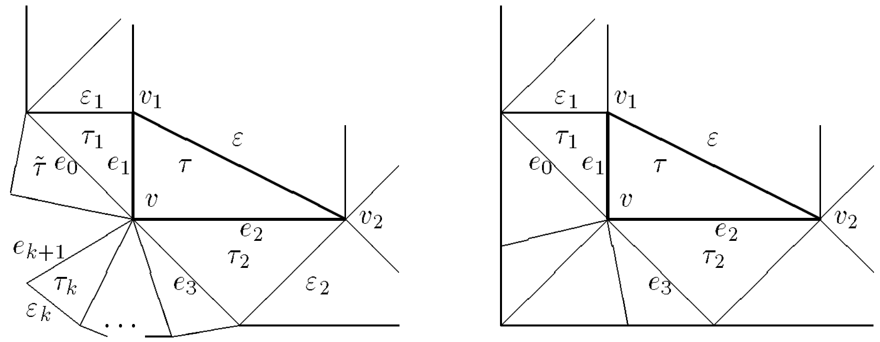
Figure 9.

<u>Case 3</u>: τ shares two edges with $\mathcal{T}$.

This is the only difficult case, because here it is the situation that nodal variables are lost when going from $\mathcal{T}$ to $\mathcal{T}'$. Let e_1 and e_2 denote the edges that τ shares with $\mathcal{T}$, and let v be the common vertex of e_1 and e_2. If v is a nonsingular interior vertex of $\mathcal{T}'$ (i.e., $\ell_{\mathcal{T}'}(v) = 1$), then one of the edge–node variables (type iii) for $\mathcal{T}$ must be removed when τ is added to form $\mathcal{T}'$. If $\ell_{\mathcal{T}'}(v) \leq 0$, then one nodal variable for $\mathcal{T}$ of type (ii) must be removed. What will be demonstrated is that the basis functions in $S_4(\mathcal{T})$ corresponding to either of these nodes can not be extended in a C^1 fashion to Π'. Thus any function in $S_4(\mathcal{T}')$ must automatically have that component zero.

For simplicity, we begin with the case $\ell_{\mathcal{T}'}(v) \leq 0$. The nodal variables for $\mathcal{T}'$ contain a set of valid nodal variables for $\mathcal{T}$, except for cross derivatives at the vertices of τ. If v_1 and v_2 are the other vertices of τ, then the argument at the end of Cases 2 &

4 shows that it suffices to assume that the τ-cross derivatives at v_1 and v_2 are not nodal variables for $\mathcal{T}'$. Thus it is only necessary to consider cross derivatives at v. Suppose initially that the nodal variable of type (ii) for $\mathcal{T}'$ at v (corresponding to the component containing $\tau \setminus \{v\}$ in $\mathrm{star}_{\mathcal{T}'}(v) \setminus \{v\}$) is the τ-cross derivative, and suppose that $\varphi \in S_4(\mathcal{T}')$ has vanishing nodal values. If τ_1 and τ_2 are the triangles in $\mathcal{T}$ having e_1 and e_2, respectively, as an edge, then the argument given at the end of Cases 2 and 4 shows that the τ_i cross derivatives of φ at v vanish, $i = 1, 2$, as well. Thus a valid set of nodal variables for $\mathcal{T}$ vanishes for φ, and hence $\varphi\,|_{\Pi} \equiv 0$. This implies that φ vanishes to second order on both e_1 and e_2, and since φ also vanishes at the midpoint of the remaining edge of τ, $\varphi\,|_{\tau} \equiv 0$ as well.



Non–interior vertex, v. Nonsingular, interior vertex, v.

Figure 10. Notation for Case 3.

Now suppose that the τ-cross derivative at v is not a nodal variable for $\mathcal{T}'$, but rather the $\tilde{\tau}$-cross derivative at v is one, for $\tilde{\tau} \setminus \{v\}$ contained in the component of $\tau \setminus \{v\}$ in $\mathrm{star}_{\mathcal{T}'}(v) \setminus \{v\}$. Suppose (without loss of generality) that $\tilde{\tau} \setminus \{v\}$ and $\tau_1 \setminus \{v\}$ lie in the same component of $\mathrm{star}_{\mathcal{T}}(v) \setminus \{v\}$, and that $\phi \in S_4(\mathcal{T}')$ has all nodal values zero. Since the $\tilde{\tau}$-cross derivative at v is a valid nodal variable for $\mathcal{T}$, we conclude that $\phi\,|_{\Pi} = c\psi$ where ψ is the basis function for the τ_2-cross derivative at v because all other $\mathcal{T}$ nodal variables of ϕ are known to vanish. We shall show that ψ can not be extended as a C^1 function to Π'. Since $\varphi \in S_4(\mathcal{T}')$ and ψ can not be extended to be in $S_4(\mathcal{T}')$, c must be zero. Thus $\varphi\,|_{\Pi} \equiv 0$, and the argument proceeds as above. Thus it remains to show that ψ can not be extended.

Our technique is simply to construct ψ. Let $\tau_3, \tau_4, \ldots, \tau_k$ be the remaining triangles such that $\tau_j \setminus \{v\}$ is in the component of $\tau_2 \setminus \{v\}$ in $\mathrm{star}_{\mathcal{T}}(v) \setminus \{v\}$, $j = 3, \ldots, k$. Here we assume they are ordered so that τ_j and τ_{j-1} share an edge e_j, $j = 3, \ldots, k$. Let e_{k+1} be the other edge of τ_k having v as a vertex. Let ε_j be the remaining edge of τ_j, $j = 2, \ldots, k$, i.e., the one not containing v. (See Figure 10.) For any edge e, let L_e denote the linear function of slope one vanishing on e. Define χ on $\tau_2 \cup \cdots \cup \tau_k$ via

$$\chi\,|_{\tau_j} = c_j L_{e_j} L_{e_j+1} L_{\varepsilon_j}^2, \quad j = 2, \ldots, k, \tag{7.9}$$

for some constants c_j. A computation similar to (7.8) shows that the τ_2-cross derivative of χ at v is a nonzero multiple of c_2:

$$\partial_{e_2}\partial_{e_3}(\chi\,|_{\tau_2})(v) = c_2(\partial_{e_2} L_{e_3})(\partial_{e_3} L_{e_2}) L_{\varepsilon_2}(v)^2.$$

Thus $c_2 \neq 0$ can be chosen so that the τ_2-cross derivative of $(\psi - \chi)$ at v is zero. The function χ is clearly continuous on $\tau_2 \cup \cdots \cup \tau_k$, since it vanishes on each e_j, $2 \leq j \leq k$. Furthermore, since χ vanishes to third order at each vertex $\varepsilon_j \cap \varepsilon_{j+1}$, $j = 2, \ldots, k-1$, and to second order at v, the slope of χ normal to e_j vanishes to second order at one end of e_j and to first order at the other, $j = 2, \ldots, k$. The constants c_j, $j = 3, \ldots, k$, can be chosen so that the slope of

$$\chi \mid_{\tau_{j-1}} - \chi \mid_{\tau_j}$$

normal to e_j vanishes at the midpoint of e_j, $j = 3, \ldots, k$. Notice that, by induction on j, all the c_j's are nonzero. With the constants $\{c_j\}$ chosen in this way, χ is C^1 in $\tau_2 \cup \cdots \cup \tau_k$. Since it vanishes to second order at v and on the edges ε_j, $j = 2, \ldots, k$, χ can be extended by zero as a function in $S_4(\mathcal{T})$. But all nodal values for $\mathcal{T}$ of $\chi - \psi$ are zero by construction, thus $\chi = \psi$. To prove that ψ can not be extended to a function in $S_4(\mathcal{T}')$, Lemma 1 implies that we simply have to check that

$$\partial_{e_1}\partial_{e_2}(\psi \mid_{\tau_2})(v) \neq 0,$$

since ψ vanishes identically in τ_1. But

$$\partial_{e_1}\partial_{e_2}(\psi \mid_{\tau_2})(v) = c_2 \partial_{e_1}\partial_{e_2}(L_{e_2} L_{e_3} L_{\varepsilon_2}^2)(v) = c_2 (\partial_{e_1} L_{e_2})(\partial_{e_2} L_{e_3}) L_{\varepsilon_2}(v)^2 \neq 0$$

(cf. the discussion following 7.8). This completes the induction step for Case 3 provided that $\ell_{\mathcal{T}'}(v) \leq 0$.

Now suppose that $\ell_{\mathcal{T}'}(v) = 1$. If v is a singular vertex, then no edge–node variables must be removed from the set of variables for $\mathcal{T}$. The argument at the end of Cases 2 & 4 shows that it suffices to assume that none of the τ-cross derivatives is a nodal variable for $\mathcal{T}'$. Thus the set of nodal variables for $\mathcal{T}'$ contains a valid set for $\mathcal{T}$, and arguments given previously in the case $\ell_{\mathcal{T}'}(v) \leq 0$ may be repeated to complete the induction step in this case. Thus it remains to consider the case $\ell_{\mathcal{T}'}(v) = 1$, with v nonsingular.

Let τ_i, $i = 1, 2$, be the triangle in $\mathcal{T}$ having e_i as an edge, and let $e_{i+(-1)^i}$ be the remaining edge of τ_i (see Figure 10). Since $\ell_{\mathcal{T}'}(v) = 1$ and $\ell_{\mathcal{T}}(v) = 0$, one edge node for $\mathcal{T}$ must be removed in forming nodal variables for $\mathcal{T}'$. The obvious candidates are the ones for e_i, $i = 1, 2$. Note that since neither vertex of e_i is interior in $\mathcal{T}$ (for both $i = 1$ and 2), neither e_1 or e_2 can have been omitted from the nodal variables for $\mathcal{T}$. Suppose that e_2 and e_0 are not parallel, and let ε_1 denote the remaining edge of τ_1. Define $\psi \in S_4(\mathcal{T})$ via

$$\psi = \begin{cases} L_{e_0}^2 L_{\varepsilon_1}^2 & \text{in } \tau_1 \\ 0 & \text{outside } \tau_1. \end{cases}$$

Then $\psi \neq 0$ at the midpoint of e_1, and all other $\mathcal{T}$ nodal variables for ψ vanish. Hence, the basis function for the e_1 edge–node variable for $\mathcal{T}$ is $c\psi$ for some (nonzero) constant c.

To construct a set of nodal variables for $\mathcal{T}'$, take a set for $\mathcal{T}$ and remove the e_1 edge–node variable. If $\varphi \in S_4(\mathcal{T}')$ vanishes at these nodal variables, then $\varphi \mid_\Pi = c\psi$ for some constant c. Thus to show that $\varphi \mid_\Pi \equiv 0$, it suffices to show that ψ can not be extended in a C^1 fashion to Π' (i.e., as an element of $S_4(\mathcal{T}')$). Once we see that $\varphi\,|_{\Pi} \equiv 0$, it follows that $\varphi\,|_{\Pi'} \equiv 0$ as well via arguments given previously.

To see that ψ can not extend to Π', observe that Lemma 1 reduces the problem to showing that

$$0 \neq \partial_{e_1}\partial_{e_2}(\psi\,|_{\tau_1} - \psi\,|_{\tau_2})(v) = \partial_{e_1}\partial_{e_2}(L^2_{e_0}L^2_{\varepsilon_1})(v) = (\partial_{e_1}L_{e_0})(\partial_{e_2}L_{e_0})L_{\varepsilon_1}(v)^2.$$

This follows because we have assumed that e_2 and e_0 are not parallel.

The above argument shows that there is a set of nodal variables for $\mathcal{T}'$ that does not include any τ-cross derivatives. We now show how such cross derivatives can be allowed. Let v_i be the other vertex of e_i, $i = 1, 2$. Then since the e_2 edge–node is a nodal variable for $\mathcal{T}'$, the argument at the end of Cases 2 & 4 shows that one can allow the τ-cross derivatives at both v and v_2 as nodal variables for $\mathcal{T}'$. However, it is not so easy to see that the τ-cross derivative at v_1 may be a nodal variable. The consideration of this case is as follows.

The τ_1-cross derivative at v_1 may be chosen as a nodal variable for $\mathcal{T}'$, and since $\ell_{\mathcal{T}'}(v_1) \leq 0$ the basis function ψ corresponding to this nodal variable satisfies

$$\psi\,|_{\tau} = cL_{e_1}L_{\varepsilon}L^2_{e_2}$$

for some $c \neq 0$ (see 7.9 and the accompanying remarks), where ε is the edge (in $\mathcal{T}'$) connecting v_1 and v_2. If $\varphi \in S_4(\mathcal{T}')$ has vanishing nodal values except for one of type (ii) at v_1 corresponding to the component of $\tau \setminus \{v_1\}$ in $st_{\mathcal{T}'}(v_1) \setminus \{v_1\}$, then the arguments given so far show that $\varphi = c'\psi$ for some constant c'. But the τ-cross derivative of ψ at v_1 is nonzero:

$$\partial_{e_1}\partial_{\varepsilon}\psi\,|_{\tau}(v_1) = c(\partial_{e_1}L_{\varepsilon})(\partial_{\varepsilon L_{e_1}})L_{e_2}(v_1)^2 \neq 0. \tag{7.10}$$

Thus vanishing of the τ-cross derivative of φ at v_1 implies that $\varphi \equiv 0$. Therefore, the τ-cross derivative at v_1 is a valid nodal variable for $\mathcal{T}'$.

The above arguments complete the induction step when the edges e_2 and e_0 are not parallel. If these edges happen to be parallel, but e_1 and e_3 are not parallel, then the same argument works simply via a switch in notation. On the other hand, it could happen that both $e_2 \parallel e_0$ and $e_1 \parallel e_3$. In this case, the vertex v is quasi–singular at τ. We have simply assumed that no such vertices occur, so we have completed the proof of Theorem 5.

The arguments above fail when v is quasi–singular at τ, and extending the results to meshes with quasi–singular vertices remains open.

References

[1] Alfeld, P., B. Piper and L. L. Schumaker, *An explicit basis for C^1 quartic bivariate splines,* **SIAM J. Num. Anal. 24** (1987), pp. 891-911.

[2] Bergman, S., and M. Schiffer, **Kernel Functions and Elliptic Differential Equations in Mathematical Physics,** Academic Press, 1953.

[3] Billera, L. J., *Homology of smooth splines: generic triangulations and a conjecture of Strang,* **Trans. AMS 310** (1988), pp. 325-340.

[4] de Boor, C., *B-Form basics,* in **Geometric Modeling: Algorithms and New Trends**, G. E. Farin, ed., Philadelphia: SIAM, 1987, pp. 131-148.

[5] de Boor, C., private communication.

[6] de Boor, C., and K. Höllig, *Approximation order from bivariate C^1-cubics: a counterexample,* **Proc. AMS 97** (1983), pp. 649-655.

[7] Chui, C. K., **Multivariate Splines**, Philadelphia: SIAM, 1988.

[8] Ciarlet, P. G., **The Finite Element Method for Elliptic Problems**, Amsterdam: North Holland Publishing Company, 1978.

[9] Morgan, J., and R. Scott, *A nodal basis for C^1 piecewise Polynomials of degree $n \geq 5$,* **Math. of Comp. 29** (1975), pp. 736-740.

[10] Powell, M. J. D., *Piecewise quadratic surface fitting for contour plotting,* in **Software for Numerical Mathematics**, D. J. Evans, ed., London: Academic, 1974, pp. 253-271.

[11] Scott, R., *C^1 continuity via constraints for 4th Order Problems,* in **Mathematical Aspects of Finite Elements in Partial Differential Equations**, C. de Boor, ed., New York: Academic Press, 1974, pp. 171-193.

[12] Scott, L. R., and M. Vogelius, *Norm estimates for a maximal right inverse of the divergence operator in spaces of piecewise polynomials,* $\mathbf{M^2AN}$ (formerly **R.A.I.R.O. Analyse Numérique**) **19** (1985), pp. 111-143.

[13] Scott, L. R., and M. Vogelius, *Conforming finite element methods for incompressible and nearly incompressible continua,* **Large Scale Computations in Fluid Mechanics**, **Lectures in Applied Math. 22 (Part 2)** (1985), B. E. Engquist, *et al.*, eds., Providence: AMS, pp. 221-244.

[14] Strang, G., *The dimension of piecewise polynomials, and one-sided approximation,* in **Conf. on Numerical Solution of Differential Equations, Lecture Notes in Mathematics No. 363**, G. A. Watson, ed., Berlin: Springer-Verlag, 1974, pp. 144-152.

[15] Strang, G., *Piecewise polynomials and the finite element method,* **Bull. AMS 79** (1973), pp. 1128-1137.

[16] Strang, G., and G. J. Fix, **Analysis of the Finite Element Method,** Prentice-Hall, 1973.

[17] W. Whitely, *A Matrix for splines,* **J. Approx. Th.**, to appear.

[18] Zienkiewicz, O. C., **The Finite Element Method in Engineering Science**, New York: McGraw–Hill, 1971.