17.1. DISCONTINUOUS COEFFICIENTS 185

There is a subtle dependence of the regularity of the solution in the case of discontinuous coefficients [219]. It is not in general the case that the gradient of the solution is bounded. However, from the variational derivation, we see that the gradient of the solution is always square integrable. A bit more is true, that is, the $p$-th power of the solution is integrable for $2 \leq p \leq P_{\varepsilon}$ where $P_{\varepsilon}$ is a number bigger than two depending only on the ellipticity constant $\varepsilon$ in (17.7) (as $\varepsilon$ tends to zero, $P_{\varepsilon}$ tends to two).

17.1.1 Coercivity and continuity

The assumption (17.5) immediately implies coercivity of the bilinear form (17.3). For each $x \in \Omega$, we take $\xi_i = v, i(x)$ and apply (17.5):

$$\int_{\Omega} |\nabla v(x)|^2 \, dx = \int_{\Omega} \sum_{i=1}^{d} v, i(x)^2 \, dx \leq c_0 \int_{\Omega} \sum_{i,j=1}^{d} \alpha_{ij}(x)v, i(x)v, j(x) \, dx = c_0 a_{\alpha}(v, v). \quad (17.8)$$

Similarly, (17.6) implies that the bilinear form (17.3) is bounded:

$$a_{\alpha}(u, v) = \int_{\Omega} \sum_{i,j=1}^{d} \alpha_{ij}(x)u, i(x)v, j(x) \, dx \leq c_1 \int_{\Omega} |\nabla u(x)||\nabla v(x)| \, dx \leq c_1 ||u||_{H^1(\Omega)} ||v||_{H^1(\Omega)}, \quad (17.9)$$

using the Cauchy-Schwarz inequality (3.15).

17.1.2 Flux continuity

Using the variational form (17.3) of the equation (17.1), we will see that the flux

$$\sum_{i=1}^{d} \alpha_{ij}(x) \frac{\partial u}{\partial x_i}(x)n_j \quad (17.10)$$

is continuous across an interface normal to $n$ even when the $\alpha_{ij}$'s are discontinuous across the interface. This implies that the normal slope of the solution must have a jump (that is, the graph has a kink).

The derivation of (17.10) is just integration by parts. Suppose that $\Omega = \Omega_1 \cup \Omega_2$ and that the coefficients are smooth on the interiors of $\Omega_i$, $i = 1, 2$, but have a jump across $\Gamma = \overline{\Omega_1} \cap \overline{\Omega_2}$. Suppose that $v = 0$ on $\partial \Omega$. Define $w = v\alpha \nabla u$ and apply the divergence theorem on each $\Omega_i$ separately to get

$$\int_{\Gamma} v \mathbf{n} \cdot \alpha \nabla u \, ds = \int_{\Omega_1} \nabla \cdot \mathbf{w} \, dx = \int_{\Omega_1} (\alpha \nabla u) \cdot \nabla v \, dx + \int_{\Omega_1} v \nabla \cdot (\alpha \nabla u) \, dx$$

Summing this over $i$ and using (17.1) we get

$$\int_{\Gamma} v [\mathbf{n} \cdot \alpha \nabla u] \, ds = a(u, v) - \int_{\Omega} f v \, dx = 0, \quad (17.11)$$

\(^1\)We are being a bit picky here about whether the sets $\Omega_i$ include their boundaries (that is, are closed) or not. To write $\Omega = \Omega_1 \cup \Omega_2$, one of the $\Omega_i$'s has to include the overlap $\Gamma$.\)