Calcolo manuscript No.
(will be inserted by the editor)

A quasi-local interpolation operator preserving the discrete divergence

V. Girault*, **L.R. Scott****

Address(es) of author(s) should be given

The date of receipt and acceptance will be inserted by the editor

Contents

Abstract. We construct an operator that preserves the discrete divergence and has the same quasi-local approximation properties as a regularizing interpolant; this is very useful when discretizing nonlinear incompressible fluid models. For low-degree finite elements, such operators have an explicit expression, from which local approximation properties can be easily derived. But for higher-degree finite elements, an explicit expression is generally not available and this construction is achieved by proving a global discrete inf-sup condition while using only local arguments. We write this construction in a general case, for conforming and non-conforming elements, and then give some applications.

Mathematics Subject Classification (1991): 76D05, 65N15, 65N30, 65N55

Keywords: inf-sup condition, discrete zero divergence, $W^{m,p}$ estimates.

* Laboratoire d'Analyse Numérique, Université Pierre et Marie Curie, 75252 Paris cedex 05, France. e-mail: girault@ann.jussieu.fr

** Department of Mathematics, University of Chicago, Chicago, Ill, USA. e-mail: ridg@uchicago.edu

1. Introduction

To perform the numerical analysis of schemes approximating models of non-linear incompressible fluids, we frequently require an operator that preserves the property of discrete zero divergence and enjoys the same local approximation properties as a standard interpolation operator. Although a classical lemma of Fortin [14] states that the uniform discrete inf-sup condition for the divergence is equivalent to the existence of an approximation operator that preserves the discrete divergence, this operator is not necessarily local and is *a priori* only stable in the H^1 norm. Therefore, we must construct our operator directly.

When dealing with low-degree finite elements, such as the Bernardi-Raugel element [5], the Crouzeix-Raviart element [13], or the mini-element [2], this operator can be easily constructed explicitly and good approximation properties can be deduced from its expression (cf. for example Crouzeix & Raviart [13], Girault & Raviart [17], Girault & Lions [16]). But in general, this construction is much more difficult as soon as the finite-elements' degree is greater than or equal to two (with the exception of the non-conforming element of degree two of Fortin & Soulié [15]).

The equivalence stated in Fortin's Lemma suggests establishing a local inf-sup condition, i.e., restricted to macro-elements. Then the corresponding operator will be quasi-local. And, since macro-elements have at most a fixed (and small) number of elements, the equivalence of norms in finite-dimensional spaces will imply approximation properties of the operator in L^p and $W^{1,p}$ norms. To this end, we shall establish a generalization of the theorem of Boland & Nicolaides [6] and Stenberg [26]. Our generalization allows on one hand overlappings of macro-elements and on the other hand it eliminates taking a linear combination with a global inf-sup condition for piecewise constant pressures.

The modification in the last step makes the proof completely local in the sense that it only requires an inf-sup condition on macro-elements. It is achieved by constructing an auxiliary approximation operator that preserves the mean-value of the divergence in each element. This can be easily done as soon as the degree of finite-elements is at least two (in two dimensions, and three in three dimensions) precisely the case in which we are interested. Reduction to functions whose divergence has zero mean-value in each element has already been used by Crouzeix & Falk in [12] in the case of non-conforming elements of degree three.

This article is written for simplicial elements, but it extends easily to quadrilateral elements. The operator is constructed in Section 1, in a general setting, for conforming and non-conforming finite elements in two or three dimensions. Section 2 describes some applications with continuous or discontinuous approximations of the pressure.

We shall use the following notation; for the sake of simplicity, we define the spaces in three dimensions. Let (k_1, k_2, k_3) denote a triple of non-negative integers, set $|k| = k_1 + k_2 + k_3$ and define the partial derivative ∂^k by

$$\partial^k v = \frac{\partial^{|k|} v}{\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3^{k_3}} .$$

Then, for any non-negative integer m and number $r \geq 1$, recall the classical Sobolev space (cf. Adams [1] or Nečas [24])

$$W^{m,r}(\Omega) = \big\{ v \in L^r(\Omega)\,;\, \partial^k v \in L^r(\Omega)\ \forall |k| \leq m \big\} ,$$

equipped with the seminorm

$$|v|_{W^{m,r}(\Omega)} = \Big(\sum_{|k|=m} \int_\Omega |\partial^k v|^r \, d\mathbf{x} \Big)^{1/r} , \tag{1.1}$$

and norm (for which it is a Banach space)

$$\|v\|_{W^{m,r}(\Omega)} = \Big(\sum_{0 \leq k \leq m} |v|^r_{W^{k,r}(\Omega)} \Big)^{1/r} ,$$

with the usual extension when $r = \infty$. The reader can refer to Lions & Magenes [22] and Grisvard [20] for extensions of this definition to non-integral values of m. When $r = 2$, this space is the Hilbert space $H^m(\Omega)$. The definitions of these spaces are extended straightforwardly to vectors, with the same notation, but with the following modification for the norms in the non-Hilbert case. Let $\mathbf{u} = (u_1, u_2, u_3)$; then we set

$$\|\mathbf{u}\|_{L^r(\Omega)} = \Big[\int_\Omega \|\mathbf{u}(\mathbf{x})\|^r \, d\mathbf{x} \Big]^{1/r} ,$$

where $\|\cdot\|$ denotes the Euclidean vector norm.

For functions that vanish on the boundary, we define

$$H^1_0(\Omega) = \big\{ v \in H^1(\Omega)\,;\, v|_{\partial\Omega} = 0 \big\} ,$$

and recall Poincaré's inequality: there exists a constant C such that

$$\forall v \in H^1_0(\Omega) \ , \ \|v\|_{L^2(\Omega)} \leq C \operatorname{diam}(\Omega) |v|_{H^1(\Omega)} . \tag{1.2}$$

Owing to (1.2), we use the seminorm $|\cdot|_{H^1(\Omega)}$ as a norm on $H^1_0(\Omega)$.

We shall also use the standard spaces for incompressible flows in d dimensions (for $d = 2$ or 3):

$$V = \left\{\mathbf{v} \in H^1_0(\Omega)^d\,;\ \operatorname{div}\mathbf{v} = 0 \text{ in } \Omega\right\},$$

$$V^\perp = \left\{\mathbf{v} \in H^1_0(\Omega)^d\,;\ \forall \mathbf{w} \in V\,,\ (\nabla\,\mathbf{v}, \nabla\,\mathbf{w}) = 0\right\},$$

$$L^2_0(\Omega) = \left\{q \in L^2(\Omega)\,;\ \int_\Omega q\,d\mathbf{x} = 0\right\}.$$

2. Construction of the operator

Let Ω be a Lipschitz-continuous domain in $\mathbb{R}^d$ ($d = 2$ or 3), with a polygonal or polyhedral boundary $\partial\Omega$. Let $\mathcal{T}_h$ be a regular (also called non-degenerate) family of triangulations of $\overline{\Omega}$ (cf. Ciarlet [11]): there exists a constant σ, independent of h and T, such that

$$\forall T \in \mathcal{T}_h\ ,\ \frac{h_T}{\rho_T} \le \sigma\,, \tag{2.3}$$

where h_T is the diameter of T and ρ_T is the diameter of the sphere inscribed in T. Let $k \ge 1$ be an integer and let (X_h, M_h) be a pair of finite-element spaces such that M_h is contained in $L^2_0(\Omega)$ and the restriction to each element of functions of X_h and M_h contains respectively $\mathbb{P}^d_k$ and $\mathbb{P}_{k-1}$. Here, $\mathbb{P}_k$ denotes the space of polynomials of total degree less than or equal to k in d variables. In the case of a conforming approximation, we choose X_h contained in $H^1_0(\Omega)^d$. In the case of a non-conforming approximation, we only ask that the restriction of functions of X_h to each element T belong to $W^{1,\infty}(T)^d$ with suitable weak matching conditions on inter-element boundaries and weak boundary conditions on $\partial\Omega$. This should include at least continuity of the average value

$$\int_{T'} \mathbf{v}_h\,ds\,, \tag{2.4}$$

over each element interface T' and this quantity should vanish on each face T' contained in $\partial\Omega$ (cf. [13]). Condition (2.4) is of course the simplest form of the "patch test" of Bruce Irons (cf. [11]). To take into account this possible nonconformity, for each number $p \ge 1$, we equip X_h with the broken norm:

$$[\mathbf{v}]_{W^{1,p}(\Omega)} = \Big(\sum_{T\in\mathcal{T}_h} |\mathbf{v}|^p_{W^{1,p}(T)}\Big)^{1/p}\,, \tag{2.5}$$

where the seminorm is defined in (1.1), and we denote

$$[f,g]_{\Omega} = \sum_{T\in\mathcal{T}_h} \int_T f\, g\, d\mathbf{x}$$

which will be used for functions defined only piecewise such as $\operatorname{div}\mathbf{v}$ for discontinuous $\mathbf{v}$.

Remark 1. At first sight, (2.5) defines a semi-norm, but in fact, it is a norm on X_h. Indeed, if $\mathbf{v}$ belongs to X_h and $[\mathbf{v}]_{W^{1,p}(\Omega)} = 0$, then $\mathbf{v} = \mathbf{c}_T$, a constant in each element T. But the continuity of $\int_{T'} \mathbf{v}\, ds$ over each element interface T' implies that $\mathbf{c}_T = \mathbf{c}$, a single constant independent of T. Finally, the fact that

$$\forall T' \in \partial\Omega\,,\ \int_{T'} \mathbf{v}\, ds = \mathbf{0}\,,$$

implies that $\mathbf{c} = \mathbf{0}$. $\square$

With the above spaces, norms and scalar products, we propose to construct an operator $P_h \in \mathcal{L}(H_0^1(\Omega)^d; X_h)$, satisfying:

$$\forall \mathbf{w} \in H_0^1(\Omega)^d\,,\ \forall q_h \in M_h\,,\ [q_h, \operatorname{div}(P_h(\mathbf{w}) - \mathbf{w})]_{\Omega} = 0\,, \tag{2.6}$$

$$\begin{aligned} &\forall \mathbf{v} \in W^{s,p}(\Omega)^d\,, \forall T \in \mathcal{T}_h\,,\\ |P_h(\mathbf{v}) - \mathbf{v}|_{W^{m,q}(T)} \leq\, &C_1 h_T^{s-m+d(\frac{1}{q}-\frac{1}{p})} |\mathbf{v}|_{W^{s,p}(\Delta_T)}\,, \end{aligned} \tag{2.7}$$

for all real numbers $1 \leq s \leq k+1$, $1 \leq p, q \leq \infty$, and integer $m = 0$ or 1 such that

$$W^{s,p}(\Omega) \subset W^{m,q}(\Omega)\,,$$

where the constant C_1 is independent of h and T and Δ_T is a suitable macro-element containing T. It can be easily checked that (2.6) and (2.7) with $p = q = 2$ and $m = s = 1$ imply the uniform inf-sup condition between X_h and M_h:

$$\inf_{q_h\in M_h} \sup_{\mathbf{v}_h\in X_h} \frac{[q_h, \operatorname{div}\mathbf{v}_h]_{\Omega}}{[\mathbf{v}_h]_{H^1(\Omega)} \|q_h\|_{L^2(\Omega)}} \geq \beta^*\,, \tag{2.8}$$

with a constant $\beta^* > 0$ independent of h. But, of course (2.6) and (2.7) yield much much more powerful results than (2.8).

For eliminating the step that deals with piecewise constant pressures in the proof of [6] and [26], we assume that there exists an interpolation operator $\Pi_h \in \mathcal{L}(H_0^1(\Omega)^d; X_h)$ that satisfies, for any $\mathbf{v} \in H_0^1(\Omega)^d$:

$$\forall T \in \mathcal{T}_h\,, \int_T \operatorname{div}(\Pi_h(\mathbf{v}) - \mathbf{v})\, d\mathbf{x} = 0\,, \tag{2.9}$$

and the same approximation property as P_h, namely, with the same notation:

$$\begin{aligned} &\forall \mathbf{v} \in W^{s,p}(\Omega)^d\,, \forall T \in \mathcal{T}_h\,, \\ &|\Pi_h(\mathbf{v}) - \mathbf{v}|_{W^{m,q}(T)} \leq C_2 h_T^{s-m+d(\frac{1}{q}-\frac{1}{p})} |\mathbf{v}|_{W^{s,p}(D_T)}\,, \end{aligned} \tag{2.10}$$

provided $W^{s,p}(\Omega) \subset W^{m,q}(\Omega)$, where the constant C_2 is independent of h and T and D_T is the union of all triangles that share a side or a vertex with T. We shall see in Section 2 that this operator is easily constructed as soon as $k \geq 2$, in two dimensions, and $k \geq 3$ in three dimensions. Unfortunately, we do not yet see how to extend our results to the case $k = 2$ in three dimensions.

Remark 2. Observe that (2.10) implies that

$$\operatorname{dist}(\operatorname{supp}(\Pi_h(\mathbf{v}))^c, \operatorname{supp}(\mathbf{v})) \leq C_3 h\,, \tag{2.11}$$

with a constant C_3 that is independent of h. □

Now, we shall construct P_h by correcting locally Π_h. For this, we assume that $\overline{\Omega}$ is the union of a finite set of macro-elements $\{\mathcal{O}_i\}_{i=1}^R$:

$$\overline{\Omega} = \cup_{i=1}^R \mathcal{O}_i\,, \tag{2.12}$$

such that each $\mathcal{O}_i$ is connected, is the union of elements T, and the maximum number of elements T in $\mathcal{O}_i$ is bounded by a constant L_1. We assume that $\mathcal{O}_i \neq \mathcal{O}_j$ for $i \neq j$; we do not assume that they are disjoint, but we suppose that each T can belong to at most L_2 macro-elements. This implies that the maximum number of macro-elements that can intersect a given one is bounded by a constant $L_3 \leq L_1 L_2$. Here L_1 and L_2 are independent of h.

For each function q_h in M_h, we define in each T

$$\tau(q_h)|_T = q_h - \frac{1}{|T|} \int_T q_h\, d\mathbf{x}\,.$$

Clearly $\tau(q_h)$ is a piecewise polynomial of the same degree as q_h, and

$$\tilde{M}_h = \tau M_h = \{\tau(q_h)\,;\, q_h \in M_h\}$$

is a linear space, although it is not necessarily a subspace of M_h. We also introduce the following spaces:

$$\tilde{M}_h(\mathcal{O}_i) = \{\tilde{q}_h|_{\mathcal{O}_i}\,;\ \tilde{q}_h \in \tilde{M}_h\}\ \forall i = 1, \dots, R.$$

Similarly, we define the following spaces:

$$\tilde{X}_h = \left\{\mathbf{v}_h \in X_h\,;\ \forall T \in \mathcal{T}_h\,,\ \int_T \operatorname{div} \mathbf{v}_h \, d\mathbf{x} = 0\right\},$$

$$\tilde{X}_h(\mathcal{O}_i) = \{\mathbf{v}_h \in \tilde{X}_h\,;\ \operatorname{supp}(\mathbf{v}_h) \subset \mathcal{O}_i\},$$

$$\tilde{V}_h(\mathcal{O}_i) = \{\mathbf{v}_h \in \tilde{X}_h(\mathcal{O}_i)\,;\ \forall q_h \in \tilde{M}_h(\mathcal{O}_i)\,,\ [q_h, \operatorname{div} \mathbf{v}_h]_{\mathcal{O}_i} = 0\},$$

$$\tilde{V}_h(\mathcal{O}_i)^{\perp} = \{\mathbf{v}_h \in \tilde{X}_h(\mathcal{O}_i)\,;\ \forall \mathbf{w}_h \in \tilde{V}_h(\mathcal{O}_i)\,,\ [\nabla \mathbf{v}_h, \nabla \mathbf{w}_h]_{\mathcal{O}_i} = 0\}.$$

Observe that the functions of $\tilde{M}_h$ have zero mean-value in each T. This is imposed on one hand for eliminating the piecewise constant pressures and on the other hand for allowing macro-elements overlaps. The constraint on the functions of $\tilde{X}_h$ is a compatibility condition with the functions of $\tilde{M}_h$. It will be automatically satisfied by using the operator Π_h.

Then we define P_h by

$$P_h(\mathbf{v}) = \Pi_h(\mathbf{v}) + \mathbf{c}_h(\mathbf{v})\,, \tag{2.13}$$

where $\mathbf{c}_h(\mathbf{v}) \in \tilde{X}_h$ will be constructed so that

$$\forall q_h \in \tilde{M}_h\,,\ [q_h, \operatorname{div} \mathbf{c}_h(\mathbf{v})]_{\Omega} = [q_h, \operatorname{div}(\mathbf{v} - \Pi_h(\mathbf{v}))]_{\Omega}\,. \tag{2.14}$$

If (2.9) holds, then (2.14) and the constraint on $\tilde{X}_h$ imply that P_h satisfies (2.6).

Here is the main theorem of this section.

Theorem 1. *Assume that:*

1) The triangulation satisfies (2.3) and there exists a set of macro-elements defined as above and a constant $\lambda^* > 0$, independent of h and $\mathcal{O}_i$ such that, for $1 \leq i \leq R$,

$$\inf_{q_h \in \tilde{M}_h(\mathcal{O}_i)} \sup_{\mathbf{v}_h \in \tilde{X}_h(\mathcal{O}_i)} \frac{[q_h, \operatorname{div} \mathbf{v}_h]_{\mathcal{O}_i}}{[\mathbf{v}_h]_{H^1(\mathcal{O}_i)} \|q_h\|_{L^2(\mathcal{O}_i)}} \geq \lambda^*\,. \tag{2.15}$$

2) There exists an approximation operator Π_h defined as above satisfying (2.9), (2.10).

Then there exists an operator $P_h \in \mathcal{L}(H_0^1(\Omega)^d; X_h)$ *of the form (2.13) satisfying (2.6) and*

$$\forall \mathbf{v} \in W^{s,p}(\Omega)^d\,,\ \forall 1 \le i \le R\,,$$
$$[P_h(\mathbf{v}) - \mathbf{v}]_{W^{m,q}(\mathcal{O}_i)} \le C_4 h_i^{s-m+d(\frac{1}{q}-\frac{1}{p})} |\mathbf{v}|_{W^{s,p}(\tilde{\Delta}_i)}\,, \tag{2.16}$$

for all real numbers $1 \le s \le k+1$, $1 \le p, q \le \infty$, *and integer* $m = 0$ *or* 1 *such that*

$$W^{s,p}(\Omega) \subset W^{m,q}(\Omega)\,,$$

where $\tilde{\Delta}_i$ *is a suitable macro-element with*

$$\mathrm{diam}(\tilde{\Delta}_i) \le C_5 h_i\,, \tag{2.17}$$

the constants C_4 *and* C_5 *are independent of* h *and* R *and* $h_i = \max_{T \subset \mathcal{O}_i} h_T$.

Proof. 1) In order to deal with possible macro-elements overlaps, we wish to associate a partition of $\overline{\Omega}$ to the set $\{\mathcal{O}_i\}$. To this end, we define $\Delta_1 = \mathcal{O}_1$, then we take for Δ_2 the union of all elements T that belong to $\mathcal{O}_2$, but not to Δ_1 and by induction, we choose for Δ_i the set (possibly empty) of all T that belong to $\mathcal{O}_i$ but not to $\cup_{j=1}^{i-1}\Delta_j$. (If Δ_i is empty, we simply omit it from the list.) By construction, the Δ_i are mutually disjoint,

$$\Omega = \cup_{i=1}^{R} \Delta_i\ ,\ \Delta_i \subset \mathcal{O}_i\,,\ 1 \le i \le R\,.$$

The uniform local inf-sup condition (2.15) implies that, for each i, there exists a unique function $\mathbf{c}_{h,i} \in \tilde{V}_h(\mathcal{O}_i)^\perp$ solution of

$$\forall q_h \in \tilde{M}_h(\mathcal{O}_i)\,,\ [q_h, \mathrm{div}\, \mathbf{c}_{h,i}]_{\mathcal{O}_i} = [q_h, \mathrm{div}(\mathbf{v} - \Pi_h(\mathbf{v}))]_{\Delta_i}\,. \tag{2.18}$$

(Note that $\mathbf{c}_{h,i} = \mathbf{0}$ when Δ_i is empty). Then we extend each $\mathbf{c}_{h,i}$ by zero outside $\mathcal{O}_i$ and we set

$$\mathbf{c}_h(\mathbf{v}) = \sum_{i=1}^{R} \mathbf{c}_{h,i}\,.$$

By construction, $\mathbf{c}_h(\mathbf{v}) \in \tilde{X}_h$; moreover, the support of $\mathbf{c}_{h,i}$ and the partitioning of Ω into $\{\Delta_i\}_{i=1}^R$ imply that $\mathbf{c}_h(\mathbf{v})$ satisfies (2.14). Indeed, we have

$$\begin{aligned}
[q_h, \mathrm{div}\, \mathbf{c}_h(\mathbf{v})]_\Omega &= [q_h, \mathrm{div}(\sum_{i=1}^{R} \mathbf{c}_{h,i})]_\Omega = \sum_{i=1}^{R} [q_h, \mathrm{div}\, \mathbf{c}_{h,i}]_\Omega \\
&= \sum_{i=1}^{R} [q_h, \mathrm{div}\, \mathbf{c}_{h,i}]_{\mathcal{O}_i} = \sum_{i=1}^{R} [q_h, \mathrm{div}(\mathbf{v} - \Pi_h(\mathbf{v}))]_{\Delta_i} \\
&= [q_h, \mathrm{div}(\mathbf{v} - \Pi_h(\mathbf{v}))]_\Omega\,.
\end{aligned}$$

2) The local inf-sup condition (2.15) implies that

$$[\mathbf{c}_{h,i}]_{H^1(\mathcal{O}_i)} \leq \frac{1}{\lambda^*}[\mathrm{div}(\mathbf{v} - \Pi_h(\mathbf{v}))]_{L^2(\Delta_i)}\,. \tag{2.19}$$

Let $\hat{T}$ denote the unit reference element and $\hat{\mathbf{c}}_i$ the composition of $\mathbf{c}_{h,i}|_T$ with the affine transformation that maps $\hat{T}$ onto T. Since each $\hat{\mathbf{c}}_i$ belongs to a finite-dimensional space, of dimension bounded by a fixed constant, on which all norms are equivalent, we can write for any $q \geq 2$:

$$\begin{aligned}\|\mathbf{c}_{h,i}\|_{L^q(\mathcal{O}_i)} &\leq \hat{C}\Big(\sum_{T\subset\mathcal{O}_i} |T|\|\hat{\mathbf{c}}_i\|^q_{L^2(\hat{T})}\Big)^{1/q}\\ &\leq \hat{C}\, h_i^{d/q}\Big(\sum_{T\subset\mathcal{O}_i} \|\hat{\mathbf{c}}_i\|^q_{L^2(\hat{T})}\Big)^{1/q}\\ &\leq \hat{C}\, h_i^{d/q}\Big(\sum_{T\subset\mathcal{O}_i} \|\hat{\mathbf{c}}_i\|^2_{L^2(\hat{T})}\Big)^{1/2}\\ &\leq \hat{C}\, h_i^{d/q}\rho_i^{-d/2}\|\mathbf{c}_{h,i}\|_{L^2(\mathcal{O}_i)}\,,\end{aligned} \tag{2.20}$$

where $\rho_i = \min_{T\subset\mathcal{O}_i}\rho_T$ and $\hat{C}$ denotes constants that are independent of h and i. The third inequality follows from Jensen's inequality. If $1 \leq q < 2$, Hölder's inequality and the fact that $\mathcal{O}_i$ contains at most L_1 elements give directly

$$\|\mathbf{c}_{h,i}\|_{L^q(\mathcal{O}_i)} \leq \hat{C}\, h_i^{d(1/q-1/2)}\|\mathbf{c}_{h,i}\|_{L^2(\mathcal{O}_i)}\,. \tag{2.21}$$

In the conforming case, $\mathbf{c}_{h,i} \in H^1_0(\mathcal{O}_i)^d$ and Poincaré's inequality (1.2) gives

$$\|\mathbf{c}_{h,i}\|_{L^2(\mathcal{O}_i)} \leq \hat{C}\mathrm{diam}(\mathcal{O}_i)|\mathbf{c}_{h,i}|_{H^1(\mathcal{O}_i)} \leq \hat{C}\, h_i|\mathbf{c}_{h,i}|_{H^1(\mathcal{O}_i)}\,. \tag{2.22}$$

In the non-conforming case, since $\mathcal{O}_i$ contains at most L_1 elements, the set $\{\hat{\mathbf{c}}_i\}_{T\subset\mathcal{O}_i}$ belongs to a finite-dimensional space whose dimension is bounded by a fixed constant independent of h. Let us show that

$$\Big(\sum_{T\subset\mathcal{O}_i} \|\hat{\mathbf{c}}_i\|^2_{L^2(\hat{T})}\Big)^{1/2} \quad\text{and}\quad \Big(\sum_{T\subset\mathcal{O}_i} |\hat{\mathbf{c}}_i|^2_{H^1(\hat{T})}\Big)^{1/2}$$

are two equivalent norms on this space. Indeed, if

$$\Big(\sum_{T\subset\mathcal{O}_i} |\hat{\mathbf{c}}_i|^2_{H^1(\hat{T})}\Big)^{1/2} = 0\,,$$

then each $\hat{\mathbf{c}}_i$ is constant and therefore $\mathbf{c}_{h,i}$ is constant in each T. Hence the argument of Remark 1, applied in $\mathcal{O}_i$ shows that $\mathbf{c}_{h,i} = \mathbf{0}$ in $\mathcal{O}_i$. Therefore, each $\hat{\mathbf{c}}_i = \mathbf{0}$. Thus, there exists a constant $\hat{C}$, such that

$$\begin{aligned}\Big(\sum_{T\subset\mathcal{O}_i} \|\hat{\mathbf{c}}_i\|^2_{L^2(\hat{T})}\Big)^{1/2} &\le \hat{C}\Big(\sum_{T\subset\mathcal{O}_i} |\hat{\mathbf{c}}_i|^2_{H^1(\hat{T})}\Big)^{1/2} \\ &\le \hat{C} h_i \rho_i^{-d/2} [\mathbf{c}_{h,i}]_{H^1(\mathcal{O}_i)}\,.\end{aligned} \tag{2.23}$$

In either case, when substituting (2.22) or (2.23) and (2.19) into (2.20) or (2.21), we derive for any $q \ge 1$:

$$\begin{aligned}\|\mathbf{c}_{h,i}\|_{L^q(\mathcal{O}_i)} &\le \hat{C}\, h_i^{1+d/q} \rho_i^{-d/2} [\mathbf{c}_{h,i}]_{H^1(\mathcal{O}_i)} \\ &\le \frac{\hat{C}}{\lambda^*} h_i^{1+d/q} \rho_i^{-d/2} [\mathrm{div}(\mathbf{v} - \Pi_h(\mathbf{v}))]_{L^2(\Delta_i)}\,.\end{aligned} \tag{2.24}$$

A similar argument, somewhat simpler because there is no need for Poincaré's inequality, yields for $q \ge 2$:

$$[\mathbf{c}_{h,i}]_{W^{1,q}(\mathcal{O}_i)} \le \frac{\hat{C}}{\lambda^*} h_i^{d/q} \rho_i^{-d/2} [\mathrm{div}(\mathbf{v} - \Pi_h(\mathbf{v}))]_{L^2(\Delta_i)}\,, \tag{2.25}$$

and for $1 \le q < 2$:

$$[\mathbf{c}_{h,i}]_{W^{1,q}(\mathcal{O}_i)} \le \frac{\hat{C}}{\lambda^*} h_i^{d/q - d/2} [\mathrm{div}(\mathbf{v} - \Pi_h(\mathbf{v}))]_{L^2(\Delta_i)}\,. \tag{2.26}$$

3) The expression of $\mathbf{c}_h$ gives:

$$\|\mathbf{c}_h\|_{L^q(\mathcal{O}_i)} = \Big(\int_{\mathcal{O}_i} \|\sum_{j=1}^{R} \mathbf{c}_{h,j}\|^q \, d\mathbf{x}\Big)^{1/q},$$

where $\|\cdot\|$ denotes the Euclidean norm. But since $\mathbf{c}_{h,j}$ vanishes outside $\mathcal{O}_j$, the above sum runs over all indices j such that $\mathcal{O}_j$ intersects $\mathcal{O}_i$. Let us number these indices from 1 to $R_i \le L_3$. Thus the sum on j has at most L_3 terms. Hence

$$\begin{aligned}\|\mathbf{c}_h\|_{L^q(\mathcal{O}_i)} &\le L_3^{\alpha} \Big(\int_{\mathcal{O}_i} \sum_{j=1}^{R_i} \|\mathbf{c}_{h,j}\|^q \, d\mathbf{x}\Big)^{1/q} \\ &\le L_3^{\alpha} \Big(\sum_{j=1}^{R_i} \|\mathbf{c}_{h,j}\|^q_{L^q(\mathcal{O}_i\cap\mathcal{O}_j)}\Big)^{1/q},\end{aligned} \tag{2.27}$$

where $\alpha = 1/2$ if $1 \le q < 2$, $\alpha = 1/q'$, $1/q + 1/q' = 1$, if $q \ge 2$. Hence (2.24) implies

$$\|\mathbf{c}_h\|_{L^q(\mathcal{O}_i)} \le \frac{\hat{C}}{\lambda^*}\Big(\sum_{j=1}^{R_i} h_j^{q(1+d/q)} \rho_j^{-qd/2} [\operatorname{div}(\mathbf{v} - \Pi_h(\mathbf{v}))]^q_{L^2(\Delta_j)}\Big)^{1/q},$$

where the index j is such that $\mathcal{O}_j$ intersects $\mathcal{O}_i$. Therefore, the local quasi-uniformity of $\mathcal{T}_h$ (cf. for instance [4]) and Jensen's inequality if $q \ge 2$ or Hölder's inequality if $q < 2$ yield:

$$\|\mathbf{c}_h\|_{L^q(\mathcal{O}_i)} \le \hat{C} h_i^{1+d(1/q-1/2)} [\operatorname{div}(\mathbf{v} - \Pi_h(\mathbf{v}))]_{L^2(D_i)}, \tag{2.28}$$

where D_i is the union of Δ_j for all j such that $\mathcal{O}_j$ intersect $\mathcal{O}_i$. Similarly, we derive from (2.25):

$$[\mathbf{c}_h]_{W^{1,q}(\mathcal{O}_i)} \le \hat{C} h_i^{d(1/q-1/2)} [\operatorname{div}(\mathbf{v} - \Pi_h(\mathbf{v}))]_{L^2(D_i)}. \tag{2.29}$$

Then (2.16) follows from (2.28) or (2.29) and (2.10) with $m = 1$ and $q = 2$.

Of course, if we integrate over Ω instead of $\mathcal{O}_i$, we obtain for $m = 0$ or 1:

$$[\mathbf{c}_h]_{W^{m,q}(\Omega)} \le \hat{C} h^{1-m+\min(0,d(1/q-1/2))} [\operatorname{div}(\mathbf{v} - \Pi_h(\mathbf{v}))]_{L^2(\Omega)}.$$

Remark 3. Note that

$$\operatorname{dist}\big(\operatorname{supp}(P_h(\mathbf{v}))^c, \operatorname{supp}(\mathbf{v})\big) \le C_6 h, \tag{2.30}$$

where the constant C_6 is independent of h. Indeed, if $\Delta_{i_1} \cup \cdots \cup \Delta_{i_k}$ is the union of all sets where $\operatorname{div}(\mathbf{v} - \Pi_h(\mathbf{v}))$ is not identically zero, then the support of $\mathbf{c}_h$ is contained in $\Delta_{i_1} \cup \cdots \cup \Delta_{i_k}$. Since each macro-element in this union contains at least one element where $\operatorname{div}(\mathbf{v} - \Pi_h(\mathbf{v}))$ does not vanish, the distance between the supports of $\mathbf{c}_h$ and $\operatorname{div}(\mathbf{v} - \Pi_h(\mathbf{v}))$ is smaller than the largest diameter of the macro-elements. Then (2.11) and the assumptions on the macro-elements imply (2.30). □

We end this section with an alternate proof of the inf-sup condition (2.8) that does not use macro-elements.

Theorem 2. *The inf-sup condition (2.8) holds with a constant $\beta^* > 0$ independent of h if the following hold.*

1) The triangulation satisfies (2.3).

2) There exists an approximation operator Π_h defined as above satisfying (2.9), (2.10).

3) There exists a constant $\gamma^* > 0$, independent of h, such that

$$\inf_{q_h \in \tilde{M}_h} \sup_{\mathbf{v}_h \in \tilde{X}_h} \frac{[q_h, \operatorname{div} \mathbf{v}_h]_\Omega}{[\mathbf{v}_h]_{H^1(\Omega)} \|q_h\|_{L^2(\Omega)}} \geq \gamma^* . \tag{2.31}$$

Proof. The proof is based on a global perturbation of Π_h analogous to (2.18). For $\mathbf{v} \in H_0^1(\Omega)^d$, let us solve the problem: Find $\mathbf{b}_h(\mathbf{v}) \in \tilde{V}_h(\Omega)^\perp$ solution of

$$\forall q_h \in \tilde{M}_h \, , \; [q_h, \operatorname{div} \mathbf{b}_h(\mathbf{v})]_\Omega = [q_h, \operatorname{div}(\mathbf{v} - \Pi_h(\mathbf{v}))]_\Omega \, . \tag{2.32}$$

By virtue of (2.31), this problem has a unique solution $\mathbf{b}_h(\mathbf{v}) \in \tilde{V}_h(\Omega)^\perp$, the dependence of $\mathbf{b}_h$ on $\mathbf{v}$ is linear and

$$[\mathbf{b}_h(\mathbf{v})]_{H^1(\Omega)} \leq \frac{1}{\gamma^*} [\operatorname{div}(\mathbf{v} - \Pi_h(\mathbf{v}))]_{L^2(\Omega)} \, .$$

Furthermore, (2.9) and the definition of $\tilde{X}_h$ imply that (2.32) is valid for all q_h in M_h. Finally, consider the following operator $Q_h \in \mathcal{L}(H_0^1(\Omega)^d; X_h)$:

$$Q_h(\mathbf{v}) = \Pi_h(\mathbf{v}) + \mathbf{b}_h(\mathbf{v}) \, . \tag{2.33}$$

The above arguments show that Q_h satisfies (2.6) and

$$[Q_h(\mathbf{v})]_{H^1(\Omega)} \leq [\Pi_h(\mathbf{v})]_{H^1(\Omega)} + \frac{1}{\gamma^*} [\operatorname{div}(\mathbf{v} - \Pi_h(\mathbf{v}))]_{L^2(\Omega)} \, ,$$

and in turn, this implies (2.8).

3. Applications

Let us begin by defining the classical conforming (continuous) Lagrange finite elements of degree k, denoted by $\mathcal{L}_h^k(\mathcal{T}_h)$:

$$\mathcal{L}_h^k(\mathcal{T}_h) = \left\{ v_h \in \mathcal{C}^0(\overline{\Omega}) \, ; \, \forall T \in \mathcal{T}_h \, , \, v_h|_T \in I\!\!P_k \right\} . \tag{3.34}$$

The Taylor-Hood element with continuous pressures (cf. Hood & Taylor [21]) for $k \geq 2$ in d dimensions is based on the spaces

$$X_h = \mathcal{L}_h^k(\mathcal{T}_h)^d \cap H_0^1(\Omega)^d \;\text{ and }\; M_h = \mathcal{L}_h^{k-1}(\mathcal{T}_h) \cap L_0^2(\Omega) \, . \tag{3.35}$$

In the two-dimensional case, the inf-sup condition for the pair of spaces (3.35) was established for $k = 2$ by Bercovier & Pironneau [3], then by Verfürth [27], and by [17]; this last reference gives a proof with a semi-local argument based on the approach of [6] and [26], and under the assumption that $\mathcal{T}_h$ is non-degenerate and each triangle T

has at most one edge on $\partial\Omega$. The inf-sup condition for $k = 3$ was established by Brezzi & Falk [9] (cf. also Brezzi & Fortin [10]) under these assumptions as well. Finally, Boffi has presented proofs of the inf-sup condition in two [7] and three [8] dimensions for arbitrary degree $k \geq 2$. But none of these references propose an approximation operator satisfying (2.6) and (2.7). Reference [18] proves the analogue of Lemma 1 and Theorem 1 with $k = 3$ in two dimensions, and we develop this approach more generally here.

3.1. Taylor-Hood in two dimensions

The preliminary operator Π_h^2 is derived from Scott & Zhang [25] as follows. Let T be a triangle with vertices $\mathbf{a}_i$, and opposite sides f_i, $1 \leq i \leq 3$. A polynomial p of degree two is uniquely determined in T by the six values:

$$p(\mathbf{a}_i)\ ,\ \int_{f_i} p(s)\,ds\,, 1 \leq i \leq 3\,.$$

Let $\varphi_{\mathbf{a}_i} \in I\!P_2$ and $\varphi_{f_i} \in I\!P_2$ be the Lagrange basis functions associated with these values, i.e.

$$\varphi_{\mathbf{a}_i}(\mathbf{a}_j) = \delta_{i,j}\ ,\ \int_{f_k} \varphi_{\mathbf{a}_i}(s)\,ds = 0\,, 1 \leq j,k \leq 3\,,$$
$$\varphi_{f_i}(\mathbf{a}_k) = 0\ ,\ \int_{f_j} \varphi_{f_i}(s)\,ds = \delta_{i,j}\,, 1 \leq j,k \leq 3\,.$$

For defining Π_h^2 on $H^1(\Omega)$, we regularize the above point values as follows. With each vertex $\mathbf{a}_i$, we choose once and for all a side κ_i of $\mathcal{T}_h$ with end-point $\mathbf{a}_i$. This choice is arbitrary, with one exception: for preserving vanishing boundary values, we impose in addition that κ_i be contained in $\partial\Omega$ whenever $\mathbf{a}_i$ lies on $\partial\Omega$. Let $\psi_{\mathbf{a}_i} \in I\!P_2(\kappa_i)$ be the dual basis function on κ_i, i.e.

$$\int_{\kappa_i} \psi_{\mathbf{a}_i}(s)\varphi_b(s)\,ds = \delta_{\mathbf{a}_i,b}\,, \tag{3.36}$$

where b denotes the side κ_i itself or its two end-points. Then we replace the point-value $p(\mathbf{a}_i)$ by the degree of freedom

$$\int_{\kappa_i} p(s)\psi_{\mathbf{a}_i}(s)\,ds\,.$$

Thus we define Π_h^2 by:

$$\begin{aligned}\Pi_h^2(v)(\mathbf{x}) = &\sum_{\mathbf{a}_i\in\mathcal{S}_h}\Big(\int_{\kappa_i} v(s)\psi_{\mathbf{a}_i}(s)\,ds\Big)\varphi_{\mathbf{a}_i}(\mathbf{x})\\ &+\sum_{f\in\Gamma_h}\Big(\int_f v(s)\,ds\Big)\varphi_f(\mathbf{x})\,,\end{aligned}\tag{3.37}$$

where $\mathcal{S}_h$ denotes the set of all vertices $\mathbf{a}_i$ of $\mathcal{T}_h$ and Γ_h denotes the set of all segments f of $\mathcal{T}_h$. It stems from the above choice of degrees of freedom on the sides f and the corresponding choice of basis functions that

$$\forall f\in\Gamma_h\,,\ \int_f\big(\Pi_h^2(v)-v\big)\,ds=0\,.\tag{3.38}$$

The operator Π_h^2 can be extended to vector-valued functions by applying it to each component. Of course, (3.38) implies (2.9). Furthermore, it is easy to check that Π_h^2 is a projection, by virtue of (3.36); i.e. if v_h is globally continuous in Ω and a polynomial of $\mathbb{P}_2$ in each triangle T, then

$$\Pi_h^2(v_h)=v_h\,.$$

This property allows one to apply the argument of [25] and show that Π_h^2 satisfies the optimal approximation property (2.10) for $s\in[1,3]$.

Following Boffi [7], we make the following construction in order to verify condition (2.15). For each interior vertex σ_i, we define $\mathcal{O}_i$ to be the union of triangles in $\mathcal{T}_h$ that have σ_i as a vertex (i.e., $\mathcal{O}_i$ is the *star* of the vertex σ_i). For each $q_h\in\Pi_h$, we construct $\mathbf{v}_h$ as follows to verify (2.15).

We begin by looking at a typical edge e that has σ_i as a vertex. Since σ_i is an interior vertex, e is not a boundary edge. Choose coordinates so that e lies on the x-axis. Let T^1 and T^2 denote the two triangles in $\mathcal{T}_h$ which include e, and let e_1^j and e_2^j be the other two edges of T^j ($j=1,2$). Let ℓ_k^j denote the linear function vanishing on e_k^j whose value at the opposite vertex (on e) is one. Given $\tilde{q}_h\in\tilde{M}_h(\mathcal{O}_i)$, define

$$\mathbf{v}^e|_{T^j}=(-\ell_1^j\ell_2^j\frac{\partial\tilde{q}_h}{\partial x}|_{T^j},0)\text{ for }j=1,2.$$

Note that $\tilde{q}_h=\tau(q_h)$ has the property that

$$\frac{\partial\tilde{q}_h}{\partial x}=\frac{\partial q_h}{\partial x}$$

on the edge e, since $\tilde{q}_h - q_h$ is constant on e. Also

$$\int_{T_j} \operatorname{div} \mathbf{v}^e \, dx = 0 \text{ for } j = 1, 2,$$

by the divergence theorem, since the component of $\mathbf{v}^e$ normal to e is zero. Thus $\mathbf{v}^e$ is well defined in $\tilde{X}_h(\mathcal{O}_i)$, and by the divergence theorem,

$$\int_{T_1 \cup T_2} \operatorname{div} \mathbf{v}^e \, \tilde{q}_h \, dx = \sum_{j=1}^{2} \int_{T_j} \ell_1^j \ell_2^j \left(\frac{\partial \tilde{q}_h}{\partial x} \right)^2 dx \,.$$

Now define

$$\mathbf{v}_h = \sum_e \mathbf{v}^e,$$

where the sum is over all edges meeting at σ_i. Then

$$\int_{\mathcal{O}_i} \operatorname{div} \mathbf{v}_h \, \tilde{q}_h \, dx = \sum_{T \subset \mathcal{O}_i} \int_T \sum_{k=1}^{2} \ell_k^T \ell_0^T \left(\frac{\partial \tilde{q}_h}{\partial e_k^T} \right)^2 dx \,,$$

where the sum is over all triangles meeting at σ_i, ℓ_0^T denotes the linear function associated with the edge of T not containing σ_i and e_k^T are the other two edges, with ℓ_k^T denoting the linear function vanishing on $e_{k'}^T$ where $\{k, k'\} = \{1, 2\}$.

Using a straightforward extension of the arguments of Theorem II.4.2, pp.178,179 in [17], we can use the above construction to prove the following in two dimensions, which in particular implies (2.15).

Lemma 1. *Assume that $\mathcal{T}_h$ is non-degenerate and each $T \in \mathcal{T}_h$ has at least one interior vertex. Let the spaces X_h and M_h be defined by (3.35) for $k \geq 2$. Then for any $\mathcal{O}_i$ and for all $q_h \in \tilde{M}_h(\mathcal{O}_i)$, the function $\mathbf{v}_h$ defined above satisfies:*

$$\begin{gathered} \forall p \geq 2 \,, \ \textit{with } p' \textit{ defined by } \frac{1}{p} + \frac{1}{p'} = 1 \,, \\ \int_{\mathcal{O}_i} q_h \operatorname{div} \mathbf{v}_h \, d\mathbf{x} \geq \hat{c} \|q_h\|_{L^p(\mathcal{O}_i)} \|q_h\|_{L^{p'}(\mathcal{O}_i)} \,, \end{gathered} \tag{3.39}$$

$$\forall p \geq 2 \,, \ \|\mathbf{v}_h\|_{L^p(\mathcal{O}_i)} \leq \hat{c} h_i^{2/p} \|q_h\|_{L^2(\mathcal{O}_i)} \,, \tag{3.40}$$

$$\forall p \geq 2 \,, \ |\mathbf{v}_h|_{W^{1,p}(\mathcal{O}_i)} \leq \hat{c} \|q_h\|_{L^p(\mathcal{O}_i)} \,, \tag{3.41}$$

where $\hat{c}$ denote several constants, depending possibly on p, but independent of h, $\mathcal{O}_i$, q_h and $\mathbf{v}_h$. In these three inequalities, the exponents p are independent of each other and can be infinite.

Combining Lemma 1 with the construction of Π_h^2 proves the following.

Theorem 3. *Suppose that the dimension $d = 2$. Under the assumptions of Lemma 1, the Taylor-Hood spaces (3.35) for $k \geq 2$ have a quasi-local interpolant P_h satisfying (2.6) and (2.16).*

3.2. Taylor-Hood in three dimensions

The techniques of Boffi [8] show that Lemma 1 holds in $d = 3$ dimensions for all $k \geq 2$. On the other hand, it is not clear how to generalize the preliminary operator Π_h^2 in this case to satisfy the required property

$$\forall f \in \Gamma_h\,, \int_f \big(\Pi_h(v) - v\big)\, ds = 0\,, \tag{3.42}$$

where Γ_h denotes the set of all faces f of $\mathcal{T}_h$.

A preliminary operator Π_h^3 based on piecewise cubics can derived from Scott & Zhang [25] as in the two-dimensional case. The nodal variables are the usual vertex and edge nodal variables, together with

$$\int_{f_i} p(s)\, ds\,, 1 \leq i \leq 4$$

for the four faces f_i. Again, it follows immediately that (3.42) holds for $\Pi_h = \Pi_h^3$.

Thus we have the following result.

Theorem 4. *Suppose that the dimension $d = 3$. Under the assumptions of Lemma 1, the Taylor-Hood spaces (3.35) for $k \geq 3$ have a quasi-local interpolant P_h satisfying (2.6) and (2.16).*

3.3. Discontinuous pressure spaces

Our techniques in the case of standard conforming finite-element methods with element-wise discontinuous pressure is even simpler than the Taylor-Hood methods, because the triangles themselves can be chosen for macro-elements. For simplicity, we give an example with $k = 2$, but the argument extends very easily to arbitrary $k \geq 3$. We retain the above notation for a triangle, its vertices and sides and we introduce the three barycentric coordinates λ_i, of the vertices $\mathbf{a}_i$ defined by: $\lambda_i \in I\!P_1$ and $\lambda_i(\mathbf{a}_j) = \delta_{i,j}$ for $1 \leq i, j \leq 3$. Then, as in [17], for each triangle T, we define the polynomial space:

$$\mathcal{P}_2(T) = I\!P_2 \oplus b_T\,,$$

where $b_T = \lambda_1\lambda_2\lambda_3$ is the bubble function in T. With this notation, we choose:

$$X_h = \left\{\mathbf{v}_h \in \mathcal{C}^0(\overline{\Omega})^2\,;\, \forall T \in \mathcal{T}_h\,,\, \mathbf{v}_h|_T \in \mathcal{P}_2(T)^2\right\} \cap H_0^1(\Omega)^2\,, \quad (3.43)$$

$$M_h = \left\{q_h \in L_0^2(\Omega)\,;\, \forall T \in \mathcal{T}_h\,,\, q_h|_T \in I\!P_1\right\}. \quad (3.44)$$

This and higher-order versions of this element were introduced by [13] and Mansfield [23]. The inf-sup condition was established in [17], first locally in each triangle and then globally with the triangles as macro-elements; but no local operator was constructed. Let us describe briefly its construction.

Since $I\!P_2 \subset \mathcal{P}_2(T)$, we can define the preliminary operator $\Pi_h = \Pi_h^2$ by (3.37). It satisfies (2.9) and (2.10) for $s \in [1,3]$.

Reference [17] proves the local inf-sup condition (2.15) with $\mathcal{O}_i = T$. The function

$$\mathbf{v}_h|_T = -b_T \nabla\, q_h\,,$$

that is associated with q_h belongs indeed to $\tilde{X}_h(T)$. Hence the assumptions of Theorem 1 are satisfied and its conclusion holds.

Theorem 5. *Suppose that the dimension $d = 2$. The spaces (3.43) and (3.44) have a quasi-local interpolant P_h satisfying (2.6) and (2.16).*

3.4. Nonconforming methods

We finish with two examples of non-conforming finite-element methods. The first one, with $k = 2$, is the $I\!P_2 - I\!P_1$ element analyzed by [15]. As mentioned in the Introduction, its operator P_h can be constructed explicitly and for this reason, it does not require all the steps of Theorem 1. But for the sake of completeness, let us describe briefly its construction. We retain the above notation and on each side f_i of T, we introduce the two Gauss points $\alpha_{i,1}$ and $\alpha_{i,2}$. Next, following [15], we define the "bubble function" $b_T \in I\!P_2$:

$$b_T = 2 - 3(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)\,,$$

that takes the value 1 at the center of T and vanishes at the Gauss points $\alpha_{i,1}$, $\alpha_{i,2}$, for $1 \le i \le 3$. Then we set

$$B_h = \left\{b_h \in L^2(\Omega)\,;\, \forall T \in \mathcal{T}_h\,,\, b_h|_T = c_T b_T\,,\, c_T \in I\!R\right\},$$

that is clearly not contained in $H^1(\Omega)$, and we choose

$$X_h = \left(\mathcal{L}_h^2(\mathcal{T}_h) \cap H_0^1(\Omega)\right)^2 \oplus B_h^2\,, \quad (3.45)$$

and M_h defined by (3.44). Again, we can define $\Pi_h = \Pi_h^2$ by (3.37); it satisfies (2.9) and (2.10) for $s \in [1,3]$. Then we set

$$P_h(\mathbf{v}) = \Pi_h^2(\mathbf{v}) + \sum_{T \subset \mathcal{T}_h} \mathbf{c}_T b_T \,,$$

where $\mathbf{c}_T = (c_{T,1}, c_{T,2})$ and

$$c_{T,k} = \frac{1}{\int_T b_T(\mathbf{x})\, d\mathbf{x}} \int_T x_k \operatorname{div}(\mathbf{v} - \Pi_h^2(\mathbf{v}))\, d\mathbf{x} \,.$$

Clearly, $P_h(\mathbf{v})$ has the same support as $\Pi_h^2(\mathbf{v})$. Let B_T be the matrix of the affine transformation that maps the unit reference triangle $\hat{T}$ onto T. In view of (2.9), we easily derive that for any number $p \geq 1$ and any function $\mathbf{v} \in W^{1,p}(\Omega)$:

$$|c_{T,k}| \leq C_1 |T|^{-1/p} \|B_T\| \|\operatorname{div}(\mathbf{v} - \Pi_h^2(\mathbf{v}))\|_{L^p(T)} \,.$$

Hence, we have proved the following result.

Theorem 6. *Suppose that the dimension $d = 2$, that $\mathcal{T}_h$ is regular and that $W^{1,p}(\Omega) \subset W^{m,q}(\Omega)$, $m = 0$ or 1. The spaces (3.45) and (3.44) have a quasi-local interpolant P_h satisfying*

$$\begin{aligned} |P_h(\mathbf{v}) - \mathbf{v}|_{W^{m,q}(T)} &\leq |\Pi_h^2(\mathbf{v}) - \mathbf{v}|_{W^{m,q}(T)} \\ &\quad + C_2 h_T^{1-m+2(1/q-1/p)} \|\operatorname{div}(\Pi_h^2(\mathbf{v}) - \mathbf{v})\|_{L^p(T)} \,, \end{aligned} \tag{3.46}$$

and therefore, P_h satisfies (2.6) and (2.16) with $k = 2$.

The last example we consider is the non-conforming $\mathbb{P}_3 - \mathbb{P}_2$ element analyzed by [12] in two dimensions. Let $\alpha_{i,1}$, $\alpha_{i,2}$ and $\alpha_{i,3}$ be the three Gauss points on the side f_i of T; we choose

$$\begin{aligned} X_h = \big\{ \mathbf{v}_h \,;\, &\forall T \in \mathcal{T}_h \,,\, \mathbf{v}_h|_T \in \mathbb{P}_3^2 \,, \\ &\mathbf{v}_h|_{f_i} \text{ is continuous at } \alpha_{i,j}, 1 \leq j \leq 3, \forall f_i \subset \Gamma_h \\ &\mathbf{v}_h(\alpha_{i,j}) = \mathbf{0}, 1 \leq j \leq 3, \text{ on each side } f_i \subset \partial\Omega \big\} \,, \end{aligned} \tag{3.47}$$

$$M_h = \big\{ q_h \in L_0^2(\Omega) \,;\, \forall T \in \mathcal{T}_h \,,\, q_h|_T \in \mathbb{P}_2 \big\} \,. \tag{3.48}$$

On one hand, as the finite-element functions that are globally continuous, vanish on $\partial\Omega$, and are polynomials of degree three in each triangle belong to X_h, we can define Π_h as in [18]. This is straightforward. But on the other hand, establishing the local inf-sup condition (2.15) is much more delicate than in all the preceding examples because there seems to be no privileged arrangement of elements into macro-elements that gives a simple proof. By writing the expression of div $\mathbf{u}_h$ in each triangle T and taking into account the continuity of $\mathbf{u}_h$ at the Gauss points, Crouzeix & Falk in [12] prove (2.15) in a variety of macro-elements that can cover most commonly used meshes. There are three typical macro-elements each containing three or four triangles. And these may be augmented by a few triangles, in case these triangles do not fit into the typical macro-elements. Then the conclusion of Theorem 1 holds with $k = 3$.

References

1. R. A. Adams, *Sobolev Spaces*, Academic Press, New York, NY, 1975.
2. D. Arnold, F. Brezzi and M. Fortin, *A stable finite element for the Stokes equations*, Calcolo, **21**, 4 (1984), pp. 337–344.
3. M. Bercovier and O. Pironneau, *Error estimates for finite element method solution of the Stokes problem in the primitive variables*, Numer. Math. **33**, (1979), pp. 211–224.
4. C. Bernardi, *Optimal finite element interpolation on curved domains*, SIAM J. Numer. Anal., **26**, (1989), pp. 1212–1240.
5. C. Bernardi and G. Raugel, *Analysis of some finite elements for the Stokes problem*, Math. of Comp., **44**, 169 (1985), pp. 71–79.
6. J. Boland and R. Nicolaides, *Stability of finite elements under divergence constraints*, SIAM J. Numer. Anal., **20**, 4 (1983), pp. 722–731.
7. D. Boffi, *Stability of higher order triangular Hood-Taylor Methods for the stationary Stokes equations*, Math. Models Methods Appl. Sci. **4**, (1994), pp. 223–235.
8. D. Boffi, *Three-Dimensional Finite Element Methods for the Stokes Problem*, SIAM J. Numer. Anal., **34**, (1997), pp. 664–670.
9. F. Brezzi and R. S. Falk, *Stability of a higher order Hood-Taylor method*, SIAM J. Numer. Anal., **28**, (1991), pp. 581–590.
10. F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, New York, 1991.
11. P. G. Ciarlet, *Basic error estimates for elliptic problems - Finite Element Methods, Part 1*, Handbook of Numerical Analysis, P. G. Ciarlet and J. L. Lions, eds., North-Holland, Amsterdam, 1991.
12. M. Crouzeix and R. S. Falk, *Nonconforming finite elements for the Stokes problem*, Math. of Comp., **52**, 186 (1989), pp. 437–456.
13. M. Crouzeix and P. A. Raviart, *Conforming and non-conforming finite element methods for solving the stationary Stokes problem*, RAIRO Anal. Numér., **8**, (1973), pp. 33–76.
14. M. Fortin, *An analysis of the convergence of mixed finite element methods*, RAIRO Anal. Numér., **11**, R3 (1977), pp. 341–354.
15. M. Fortin and M. Soulié, *A non-conforming piecewise quadratic finite element on triangles*, Int. J. for Numer. Meth. in Eng., **19**, (1983), pp. 505–520.
16. V. Girault and J.-L. Lions, *Two-grid finite-element schemes for the steady Navier-Stokes problem in polyhedra*, Portug. Math., **58** 1 (2001), pp. 25–57.
17. V. Girault and P. A. Raviart, *Finite Element Methods for the Navier-Stokes Equations. Theory and Algorithms*, SCM **5**, Springer-Verlag, Berlin, 1986.

18. V. Girault and L. R. Scott, *Finite-element discretizations of a two-dimensional grade-two fluid model*, M2AN **35**, (2002), pp. 1007–1053.
19. V. Girault and L. R. Scott, *An upwind discretization of a steady grade-two fluid model in two dimensions*, to appear in Nonlinear Partial Differential Equations, Collège de France Seminar.
20. P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman Monographs and Studies in Mathematics, **24**, Pitman, Boston, MA, 1985.
21. P. Hood and C. Taylor, *A numerical solution of the Navier-Stokes equations using the finite element technique*, Comp. and Fluids 1, (1973), pp. 73–100.
22. J.-L. Lions and E. Magenes, *Problèmes aux Limites non Homogènes et Applications, I*, Dunod, Paris, 1968.
23. L. Mansfield, *Finite element subspaces with optimal rates of convergence for the stationary Stokes problem*, RAIRO Anal. Numér. **16**, 1, (1982), pp. 49–66.
24. J. Nečas, *Les Méthodes directes en théorie des équations elliptiques*, Masson, Paris, 1967.
25. L. R. Scott and S. Zhang, *Finite element interpolation of non-smooth functions satisfying boundary conditions*, Math. Comp. **54**, (1990), pp. 483–493.
26. R. Stenberg, *Analysis of finite element methods for the Stokes problem: a unified approach*, Math. Comp. **42**, (1984), pp. 9–23.
27. R. Verfürth, *Error estimates for a mixed finite element approximation of the Stokes equations*, RAIRO Anal. Numér. **18**, 2, (1984), pp. 175–182.