Stability of dispersive model equations for fluid flow

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Abstract

This article discusses the use of stability as a guide to understanding model equations for fluid flow. In particular, the end goal is to evaluate a model of fluid flow which incorporates dispersion. Such models do not have a linear stress-strain law, and hence extend models of Newtonian fluids. The model studied has been proposed in widely disparate contexts, including rheology and turbulence modeling. We indicate what stability properties have been deduced so far for both the continuum model and for numerical approximations of it.

Keywords: rheology, turbulence modeling, finite elements.

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1 Modeling philosophy

Modeling empirical phenomena has aspects of a black art. It is difficult to codify the process of modeling. We will not try to do that here, but rather observe some key components in different contexts as a guide to the

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understanding of what constitutes a good model or class of models. Our main emphasis will be on the role of stability as a guide in critiquing models.

One key ingredient in modeling is the identification of critical features that characterize interesting behavior. For example, it might be noticed that certain waves on water have interesting relationships between different length scales, such as amplitude and wave length. A second ingredient might be the formulation of an equation in a set of variables which describe this behavior in a way that respects the observed relationships. We give an example of this with regard to one-dimensional wave propagation for nonlinear, dispersive systems. Typically, the second modeling step would follow from the first, although there is no reason that it always must.

Models are by their nature only an approximation to empirical phenomena, and therefore a multitude of approximations could be equally valid from a certain point of view. However, there are certain characteristics of models that make them more interesting than others. Of course, the most basic requirement is that it provides a predictor of the empirical phenomena under study. If this did not hold, it would not be an approximation at all. But beyond this, there are notable characteristics of enduring models.

One characteristic of a "good" model is that its mathematical formulation appears as a model of a range of empirical phenomena. This can happen because the mathematical formulation has a certain structural universality. The ubiquitous appearance of the Laplace operator is one such example. Our one-dimensional example of models of nonlinear dispersive waves is another. We will argue that the three-dimensional model of dispersive fluid flow is beginning to achieve this status.

Another important characteristic of a model might be called "robustness" or perhaps even "serendipity." Here we refer to the fact that the model has exceptional ability to represent empirical phenomena to a higher degree of accuracy than might otherwise be expected. This has been observed [6] for the one-dimensional model we discuss, and it is widely felt to hold for the standard Navier-Stokes model for Newtonian fluids. But it is definitely too early to claim this for the three-dimensional model of dispersive fluid flow we discuss here. However, such a question might be an interesting litmus to apply in the future.

A final characteristic of a good model is that the mathematical formulation has appropriate stability properties. This is an important criterion for both the continuum model and for numerical approximations of it. Without stability, it is likely that models will have limited applicability. Even if the continuum model is stable in some sense, an unstable numerical approximation of it will be of little use. Of course, the first step in assessing stability should be to consider the continuum model. But a critical second step is to investigate stability of numerical approximations of it. The stability of the continuum model does not at all imply stability of any numerical approximations of it.

To summarize, we have identified four characteristics of good models, beyond the simple requirement of predicting a particular phenomenon. These are the following.

- (I) The model predicts multiple natural phenomena, which can occur in part because
- (II) the mathematical formulation has a certain structural universality.
- (III) The model has robust applicability, and
- (IV) has a stable mathematical formulation and numerical discretization.

Criterion (IV) has appeared last, but it is not at all the least in importance. It is hard to imagine a situation where (IV) did not hold and yet properties (II) and (III) did. However, (IV) is often the hardest to demonstrate, so we will postpone discussion of it as long as possible.

We begin by examining these ideas in the context of a well studied one-dimensional wave model. Subsequently, we apply the ideas gleaned from our one-dimensional example to examine a model for dispersive flow in three-dimensional fluid systems. The standard Navier-Stokes model for Newtonian fluids is not dispersive, and the model we consider may be viewed as a perturbation of it. Our primary discussion will focus on stability properties of the continuum model and of numerical approximations of it.

For simplicity, we will assume that the reader finds elsewhere introductions to the basic equations discussed here as well as to simple concepts like "dispersion" [34].

2 One-dimensional wave models

We begin by discussing some simpler models where our understanding is very well developed. An explosion of research has emanated from the observation of so-called solitary waves [30]. Solitary waves are waves whose amplitude $u(x,t)$ is described by a profile function $\phi(x)$ and whose propagation in space and time is a simple translation:

$$u(x,t) = \phi(x - ct), \tag{2.1}$$

where c is the wave speed.

Solitary waves occur in wave propagation systems having a weak nonlinearity and dispersion. The classic equation of this type is the one due to Korteweg and de Vries [22]:

$$u_t + u_x + uu_x + u_{xxx} = 0, \tag{2.2}$$

posed say for x in the whole real line $I\!R$.

Without going into the extensive history at all, let us note that a key feature of solitary waves was the observation of a relationship between the amplitude ϵ of the waves and their length $\ell = 1/\delta$, as well as the relevant time scale of order δ (required so that a wave of this length will actually go by a particular point of reference). In particular, it was observed that a critical balance should occur when the Stokes parameter

$$S = \epsilon/\delta^2 \tag{2.3}$$

is of order one. This balances the nonlinear (uu_x) and dispersive (u_{xxx}) terms in (2.2), which are then both of order ϵ smaller than the other two terms in the equation. But it also emphasizes the fact that the basic equation is the simpler linear wave propagation equation

$$u_t + u_x = 0 \tag{2.4}$$

to which nonlinear and dispersive terms have been added.

If (2.4) is the basic relationship for solitary waves, then we should be able to convert time derivatives and space derivatives ($u_t = -u_x$) essentially freely in a model like (2.2). For example, we might consider the equation of Peregrine [25] and Benjamin, Bona and Mahoney [3]

$$u_t + u_x + uu_x - u_{xxt} = 0. \tag{2.5}$$

Both (2.2) and (2.5) have solitary waves of the form (2.1) where ϕ satisfies the following ODE's:

$$(1 - c)\phi' + \phi\phi' + \phi''' = 0 \tag{2.6}$$

for (2.2) (KdV), and

$$(1-c)\phi' + \phi\phi' + c\phi''' = 0 \tag{2.7}$$

for (2.5) (PBBM). If the speed c is close to one, these equations clearly are close to each other, and it can be verified that they have similar solutions of the form

$$\phi(x) = 3\epsilon \,\mathrm{sech}^2\left(\tfrac{1}{2}\epsilon^{\frac{1}{2}} x\right) \tag{2.8}$$

for KdV, and

$$\phi(x) = 3\epsilon \,\mathrm{sech}^2\left(\tfrac{1}{2}\left(\frac{\epsilon}{1+\epsilon}\right)^{\frac{1}{2}} x\right) \tag{2.9}$$

for PBBM, with $c = 1+\epsilon$ for both equations. Moreover, one can prove rigorously, and for general initial data, that the models KdV and PBBM are equivalent over the relevant time scales of interest [7].

In the discussion above comparing the two models, we took the Stokes parameter S in (2.3) to be one. The robustness of these models for water waves can be assessed by considering the large range (at least two orders of magnitude) of S for which the (essentially equivalent) models give good predictions of water waves [6].

From a stability point of view, KdV and PBBM are quite different. The latter can be written

$$u_t = -\left(1 - \frac{\partial^2}{\partial x^2}\right)^{-1}\left(\frac{\partial}{\partial x}\right)\left(u + \tfrac{1}{2}u^2\right). \tag{2.10}$$

When we write $w = \left(1 - \frac{\partial^2}{\partial x^2}\right)^{-1} v$ we mean that w solves

$$\left(1 - \alpha\frac{\partial^2}{\partial x^2}\right) w = v \tag{2.11}$$

with w tending to zero at infinity as boundary conditions. We recognize the operator $\left(1 - \frac{\partial^2}{\partial x^2}\right)^{-1}\left(\frac{\partial}{\partial x}\right)$ as a smoothing operator (it gains one derivative), so that (2.10) (and equivalently (2.5)) may be thought of as a simple ODE (in a Hilbert space, for example). There is no stability restriction on the time step compared to the spatial discretization. Thus, for example, high-order multi-step (explicit) methods can be used effectively [8]. The stability restrictions for KdV are a much more complex story. It is now well understood how to discretize KdV accurately [31][4], but it took many years to understand how to do this.

Stability considerations alone are not the only factor in comparing the models KdV and PBBM. Another critical aspect concerns the fundamental mathematical structure, and KdV can be viewed as a completely integrable Hamiltonian system (whereas PBBM is not). One key corollary of this is that solitary wave interactions are particle-like for KdV (leading to the coining of the name *solitons*), whereas they are not for PBBM [5]. However, a simple modification of (2.5) due to Camassa and Holm [9] of the form

$$u_t + u_x + uu_x - \alpha u_{xxt} = \tfrac{2}{3}\alpha u_x u_{xx} + \tfrac{1}{3}\alpha u u_{xxx} \tag{2.12}$$

does have the soliton property. Here, α is a positive constant (which appears as α^2 in [9]) indicating the amount of dispersion; previously we just had $\alpha = 1$. From a modeling perspective, the terms on the right-hand side of (2.12) are even smaller (in terms of ϵ order) than the nonlinear and (linear) dispersive terms on the left-hand side. Thus we expect (2.12) to give equivalent results in predicting solitary waves of the sort considered here.

The CH equation (2.12) have a suggestive form

$$v_t + u_x + \tfrac{1}{3}uv_x + \tfrac{2}{3}vu_x = 0, \tag{2.13}$$

where $u = \left(1 - \alpha\frac{\partial^2}{\partial x^2}\right)^{-1} v$ (again we mean that u solves (2.11) with u tending to zero at infinity as boundary conditions). We see that (2.12) (which is a nonlinear perturbation of (2.5) and therefore also of (2.2)) can be viewed as a nonlinear advection equation for a variable v which has a simple relation to the advection velocity u. At this, we leave the one-dimensional world, but we will see the form (2.13) is the template for a three-dimensional fluid flow model including dispersion.

3 3-dimensional dispersive fluid models

For the sake of continuity, let us focus our attention on an equation of the form

$$\mathbf{v}_t - \nu\Delta\mathbf{u} + (\mathbf{curl}\,\mathbf{v}) \times \mathbf{u} + \nabla p = 0, \tag{3.14}$$

where $\mathbf{v} = (1 - \alpha\Delta)\,\mathbf{u}$, $\alpha > 0$, $\nabla \cdot \mathbf{u} = 0$ and p denotes the pressure plus other terms to which the gradient operator is applied.

If we set α to zero, we have $\mathbf{u} = \mathbf{v}$. Considering the vector identity

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{curl}\,\mathbf{u} \times \mathbf{u} + \nabla(\tfrac{1}{2}\mathbf{u} \cdot \mathbf{u}), \tag{3.15}$$

(3.14) then reduces to the Navier-Stokes equation

$$\mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{curl}\,\mathbf{u}) \times \mathbf{u} + \nabla p = 0, \tag{3.16}$$

modulo the translation that the pressure $\tilde{p} = p - \frac{1}{2}\mathbf{u} \cdot \mathbf{u}$. The nonlinear term $(\mathbf{curl}\,\mathbf{u}) \times \mathbf{u}$ may look unusual, but it differs from the usual one $\mathbf{u} \cdot \nabla\mathbf{u}$ via the transformation (3.15) and the re-definition of the "pressure" variable.

The equation (3.14) appears in the literature in multiple contexts, hence it satisfies our modeling criterion (I). But this makes it difficult to know where to start, as we do not want to favor one derivation over another. Of course, we have already done so by noting the similarity with the one-dimensional CH equation written in the form (2.13).

3.1 Rivlin-Ericksen fluids

From a historical perspective, (3.14) first appeared as a model of Rivlin and Ericksen (cf. [14]). In this approach, we begin with the description of a continuum of material with velocity $\mathbf{u}$ described by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u} = \nabla \cdot \mathbf{T}, \tag{3.17}$$

where $\mathbf{T}$ is the stress and we define the convective derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}w := w_t + \mathbf{u} \cdot \nabla w \tag{3.18}$$

for any function w (either scalar, vector or tensor valued).

In particular, we assume that the stress tensor $\mathbf{T}$ has the form (for the grade n model)

$$\mathbf{T} = -\tilde{p}\mathbf{I} + \mathcal{S}^n(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n), \tag{3.19}$$

where $\tilde{p}$ is the pressure and $\mathbf{a}_j$ are the Rivlin-Ericksen tensors defined recursively by (recall the notation (3.18))

$$\mathbf{a}_1 = \mathbf{L} + \mathbf{L}^T \quad , \quad \mathbf{L} = \nabla\,\mathbf{u}\,,$$

$$\mathbf{a}_j = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{a}_{j-1} + \mathbf{a}_{j-1}\mathbf{L} + \mathbf{L}^T\mathbf{a}_{j-1}\,,$$

$\mathcal{S}^n = \sum_{i=1}^n \mathbf{S}_i$ and each $\mathbf{S}_i$ is a polynomial in the $\mathbf{a}_j$:

$$\mathbf{S}_1 = \eta\mathbf{a}_1\ ,\ \mathbf{S}_2 = \alpha_1\mathbf{a}_2 + \alpha_2\mathbf{a}_1^2\,,$$

$$\mathbf{S}_3 = \beta_1\mathbf{a}_3 + \beta_2(\mathbf{a}_1\mathbf{a}_2 + \mathbf{a}_2\mathbf{a}_1) + \beta_3(\mathrm{tr}\,\mathbf{a}_1^2)\mathbf{a}_1\ ,\ \mathrm{etc}\,.$$

The so-called *grade-two* fluid model has the constitutive equation

$$\mathbf{T} = -\tilde{p}\mathbf{I} + \eta\mathbf{a}_1 + \alpha_1\mathbf{a}_2 + \alpha_2\mathbf{a}_1^2\,, \tag{3.20}$$

where the parameters η and α_i are material constants. Physical arguments [12][15][13] show that the coefficients should satisfy

$$\eta \geq 0\,,\quad \alpha_1 \geq 0 \quad \text{and} \quad \alpha_1 + \alpha_2 = 0\,.$$

Setting $\alpha = \alpha_1$ (and $\alpha_2 = -\alpha$) and substituting (3.20) into (3.17) yields (3.14).

For the curious, the grade-three constitutive equation has

$$\mathcal{S}^3 = \eta\mathbf{a}_1 + \alpha_1\mathbf{a}_2 + \alpha_2\mathbf{a}_1^2 + \beta_1\mathbf{a}_3 + \beta_2(\mathbf{a}_1\mathbf{a}_2 + \mathbf{a}_2\mathbf{a}_1) + \beta_3(\mathrm{tr}\,\mathbf{a}_1^2)\mathbf{a}_1\,. \tag{3.21}$$

To be consistent with thermodynamics, the conditions on the material constants for a grade-three fluid are (cf. [13])

$$\beta_1 = \beta_2 = 0,\ \beta_3 \geq 0,\ \alpha_1 \geq 0,\ \eta \geq 0 \text{ and } |\alpha_1 + \alpha_2| \leq \sqrt{24\eta\beta_3}\,. \tag{3.22}$$

3.2 Geometry of maps

When $\nu = 0$ in (3.14), we get a grade-two variant of the Euler equations. It has been known for some time that there is an intriguing geometric interpretation of flows satisfying the Euler equations. Consider the class $\mathcal{V}^s$ of vector fields $\mathbf{u}$ satisfying (a) $\mathbf{u} \in H^s(\Omega)$ for a domain (or manifold) Ω with s sufficiently large, (b) $\nabla \cdot \mathbf{u} = 0$ and (c) $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$. Given a family of vector fields $\mathbf{u}(\cdot, t) \in \mathcal{V}^s$, consider the flow $\mathbf{f} = \mathbf{f}(\mathbf{x}, t)$ generated by $\mathbf{u}$, that is, (recall (3.18))

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{f}(\mathbf{x}, t) = \mathbf{u}(\mathbf{f}(\mathbf{x}, t), t), \tag{3.23}$$

where $\mathbf{f}(\mathbf{x}, 0) = \mathbf{x}$ for all $\mathbf{x} \in \Omega$. For each t, $\mathbf{f}(\cdot, t)$ is a volume preserving diffeomorphism of Ω, the space of which we denote by $\mathcal{D}_\Omega$. Thus the map

$$t \to \mathbf{f}(\cdot, t) \tag{3.24}$$

is a curve in $\mathcal{D}_\Omega$.

We can create geometry in the set $\mathcal{D}_\Omega$ simply by choosing a metric for the tangent space. To begin with, $\mathcal{D}_\Omega$ has a natural group structure: you can compose one diffeomorphism with another to get a third. The identity for the group is the transformation $\mathcal{I}(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \Omega$. The tangent space $T_{\mathcal{D}_\Omega}(\mathcal{I})$ to $\mathcal{D}_\Omega$ at $\mathcal{I}$ can be identified with the space $\mathcal{V}^s$ of divergence-free vector fields. Thus putting an inner-product on $\mathcal{V}^s$ puts a metric on the tangent space to $\mathcal{D}_\Omega$ at $\mathcal{I}$. For example, the L^2 inner-product on $T_{\mathcal{D}_\Omega}(\mathcal{I}) \approx \mathcal{V}^s$ is given by

$$< \tau, \sigma >_{L^2} = \int_\Omega \tau(\mathbf{x}) \cdot \sigma(\mathbf{x}) \, d\mathbf{x} \tag{3.25}$$

and the H^1_α inner-product on $T_{\mathcal{D}_\Omega}(\mathcal{I}) \approx \mathcal{V}^s$ is given by

$$< \tau, \sigma >_{H^1_\alpha} = \int_\Omega \left(\tau(\mathbf{x}) \cdot \sigma(\mathbf{x}) + \alpha \nabla\tau(\mathbf{x}) : \nabla\sigma(\mathbf{x}) \right) \, d\mathbf{x}. \tag{3.26}$$

Using the group structure allows us to translate this metric to the entire tangent bundle invariantly. Remarkably [1], $\mathbf{u}$ solves the Euler equations if and only if the curve (3.24) is a geodesic in $\mathcal{D}_\Omega$ with metric (3.25) given by the L^2 inner-product on $\mathcal{V}^s$. Even more remarkably [23], $\mathbf{u}$ solves (3.14) if and only if the curve (3.24) is a geodesic in $\mathcal{D}_\Omega$ with metric (3.26) given by the H^1_α inner-product on $\mathcal{V}^s$. Being a geodesic means that this is a path of minimal length in this metric. This structural property of the model (3.14) exhibits our property (II) of a good model and makes it clear that it does not appear by chance.

3.3 Turbulence models

It is of interest to study statistical properties of ensembles of solutions of the Navier-Stokes equations (meaning (3.16) together with $\nabla \cdot \mathbf{u} = 0$). If we let $\mathbf{u}$ denote such an ensemble average of solutions, then at least one such averaged equation takes the form (3.17). Given the general considerations leading to (3.19), it is not surprising that appropriate models would lead ultimately to (3.14). A modified model, including a fourth-order dissipation term, has been shown in [10] to provide an accurate model of turbulence experiments done in a channel.

4 Stability of the grade-two model

Although we now see there could be many names for (3.14), we will stick for the moment with its first name, the grade-two model. We will always remember that $\mathbf{u}$ (and therefore $\mathbf{v} = (1 - \alpha\Delta)\mathbf{u}$) is divergence free. The first results showing the well-posedness (or stability) of these equations are due to Ouazar [24]. More precisely, he constructed a solution $\mathbf{u}$ to (3.14), with a force $\mathbf{f}$ in the right-hand side and zero on the boundary, such that

$$\|\mathbf{u}\|_{H^3} \leq C(\|\mathbf{f}\|_{L^2} + \|\mathbf{curl}\,\mathbf{f}\|_{L^2})\,.$$

From this starting point, there have been many results derived; for a survey see [11]. However, until recently, the stability results were only in spaces which required a great deal of regularity of the data and domain. Results in domains with corners were not available, and results in spaces suitable for variational (e.g., finite-element, spectral, etc.) approximation were also not proved.

There are different forms of (3.14) that can be useful in different contexts. One already used in [24] involves taking the $\mathbf{curl}$ of (3.14) and introducing the variable

$$\mathbf{z} = \mathbf{curl}\,\mathbf{v} = \mathbf{curl}\,(\mathbf{u} - \alpha\Delta\mathbf{u}). \tag{4.27}$$

This leads to a transport equation for $\mathbf{z}$:

$$\alpha\mathbf{z}_t + \nu\mathbf{z} + \alpha\mathbf{u}\cdot\nabla\mathbf{z} - \alpha\mathbf{z}\cdot\nabla\mathbf{u} = \nu\,\mathbf{curl}\,\mathbf{u}. \tag{4.28}$$

Moreover, (3.14) simplifies a bit to read

$$\mathbf{v}_t - \nu\Delta\mathbf{u} + \mathbf{z}\times\mathbf{u} + \nabla p = 0. \tag{4.29}$$

One stringent test of a model for dynamics is to see if the corresponding steady-state equations make sense. It may at first seem that this is a simpler system to consider, but in fact it encapsulates an even harder problem. The steady versions of (4.28) and (4.29) read

$$\begin{aligned} -\nu\Delta\mathbf{u} + \mathbf{z}\times\mathbf{u} + \nabla p =& 0 \\ \nabla\cdot\,\mathbf{u} =& 0 \\ \nu\mathbf{z} + \alpha\mathbf{u}\cdot\nabla\mathbf{z} - \alpha\mathbf{z}\cdot\nabla\mathbf{u} =& \nu\,\mathbf{curl}\,\mathbf{u}. \end{aligned} \tag{4.30}$$

The first two equations of this set of three equations can be recognized as a (linear) Stokes-like system. The third equation appears then as a (linear) transport equation. Of course the equations are coupled; the nonlinearity has not disappeared. Formulations of this type are often called "mixed" formulations, and the corresponding numerical discretization "mixed methods."

Having a nice name is often useful in guiding our thinking, but is not always the final solution. In fact, there are not as yet results in the 3-dimensional case for (4.30) that are fully satisfactory. However, in the 2-dimensional case, things simplify just enough for some useful stability bounds to be proved. In particular, the third component of $\mathbf{u}$ vanishes and correspondingly, $\mathbf{z}$ is of the form $(0,0,z)$, with z independent of the third coordinate, and the term $\mathbf{z}\cdot\nabla\mathbf{u}$ vanishes identically. Thus (4.30) becomes

$$\begin{aligned} -\nu\Delta\mathbf{u} + z(-u_2,u_1) + \nabla p =& 0 \\ \nabla\cdot\,\mathbf{u} =& 0 \\ \nu z + \alpha\mathbf{u}\cdot\nabla z =& \nu\,\mathrm{curl}\,\mathbf{u}, \end{aligned} \tag{4.31}$$

where now $\mathbf{u}$ denotes a 2-vector valued function and $\mathrm{curl}\,\mathbf{u} := u_{1,2} - u_{2,1}$.

The first two equations in (4.31) are well posed with $\mathbf{u}\in H^1$ as long as $z\in L^r$ for $r>1$, provided we have suitable boundary data specified for $\mathbf{u}$. More precisely, given such a z we can find a unique $\mathbf{u}\in H^1$ (and $p\in L^2$) such that

$$\begin{aligned} -\nu\Delta\mathbf{u} + (-zu_2, zu_1) + \nabla p =& 0 \\ \nabla\cdot\,\mathbf{u} =& 0 \end{aligned} \tag{4.32}$$

with appropriate bounds in terms of $\|z\|_{L^r}$. However, the third (transport) equation in (4.31) is more problematic.

Let us write the general form of the transport equation in (4.31), after dividing by ν, as

$$z + \frac{\alpha}{\nu}\mathbf{u}\cdot\nabla z = f. \tag{4.33}$$

Note that we have posed this equation with no boundary conditions; the variable z is the curl of $\mathbf{u} - \alpha\Delta\mathbf{u}$, and it would be unusual to require such data. We can show that (4.33) is well posed without boundary conditions only if the boundary conditions on $\mathbf{u}$ are such that $\mathbf{u}\cdot\mathbf{n} = 0$ on $\partial\Omega$, as we now assume. If all we know is that $\mathbf{u} \in H^1$, then $f \in L^2$ is the best we can hope for in (4.33). But then we could not hope for more than $z \in L^2$ either, as (4.33) provides no smoothing. And for $z \in L^2$ (and $\mathbf{u} \in H^1$), the term $\mathbf{u}\cdot\nabla z$ is a concern. Certainly, it will not in general be in L^2 for unrelated z and $\mathbf{u}$.

Miraculously, it is possible to show that (provided $\mathbf{u}\cdot\mathbf{n} = 0$ on $\partial\Omega$), the problem (4.33) has a unique solution $z \in L^2$ for any $f \in L^2$ [16]. In fact, you can say more, in that z lies in the space

$$X_{\mathbf{u}} := \left\{ w \in L^2 \ : \ \mathbf{u}\cdot\nabla w \in L^2 \right\} \tag{4.34}$$

and $\|z\|_{L^2} \le \|f\|_{L^2}$.

Existence of solutions to (4.33) is not difficult, but uniqueness is the hard part. It might be tempting to prove uniqueness by considering the case $f = 0$ and hoping to show that then any corresponding solution z_0 is identically zero by using an integration by parts, viz.

$$\int_\Omega w\mathbf{u}\cdot\nabla z\,dx = -\int_\Omega z\mathbf{u}\cdot\nabla w\,dx \quad \forall z, w \in X_{\mathbf{u}} \tag{4.35}$$

(recall that $\nabla\cdot\mathbf{u} = 0$ in Ω and $\mathbf{u}\cdot\mathbf{n} = 0$ on $\partial\Omega$). Applying (4.35) with $z = w = z_0$ surely implies $z_0 = 0$. However, it turns out that (4.35) and the uniqueness of the solution of (4.33) are simply equivalent. The proof of either is not simple, but it is done in [16]. The first step is to show that smooth functions are dense in the space

$$\left\{ w \in L^2 \ : \ \mathbf{u}\cdot\nabla w \in L^1 \right\}. \tag{4.36}$$

This allows us to conclude that $\|z_0\|_{L^1(\Omega)} = 0$ (when $\nabla\cdot\mathbf{u} = 0$ and $\mathbf{u}\cdot\mathbf{n} = 0$ on $\partial\Omega$; some details have been omitted). From this, we establish denseness of smooth functions in the space $X_{\mathbf{u}}$ in (4.34) provided $\nabla\cdot\mathbf{u} = 0$ and $\mathbf{u}\cdot\mathbf{n} = 0$ on $\partial\Omega$.

The fact that (4.33) has a unique solution depending boundedly on f in L^2 is exactly what we need to couple with the equations in (4.32) to show that the coupled system (4.31) is well posed [16], namely, as for the Navier-Stokes equation (3.16), it has at least one solution ($\mathbf{u}$ in H^1 and z in L^2) and each solution is bounded by the data.

The form of the stability result gives us insight into how this model functions in terms of the way it manages and transfers information. The third equation in (4.31) is a transport equation like (4.33) that moves the rotational information encoded in curl $\mathbf{u}$ along the particle paths given by the convective derivative (3.18). The fact that (4.33) can be solved, stably, without boundary conditions is at first surprising. It may be helpful to characterize the solution by the following.

Define a function ζ in the domain Ω as a function of a time-like parameter τ (τ actually parameterizes the distance backwards along particle paths), by solving

$$\frac{\partial \zeta}{\partial \tau} + \zeta + \frac{\alpha}{\nu}\mathbf{u} \cdot \nabla \zeta = 0, \tag{4.37}$$

with $\zeta(0, x) = f(x)$ for $x \in \Omega$. Taking the product of (4.37) with ζ and integrating over Ω shows that the $L^2(\Omega)$ norm of ζ must decrease exponentially in τ. Integrating (4.37) in τ shows that

$$\int_0^\infty \zeta \, d\tau = z. \tag{4.38}$$

Thus one way we could imagine approximating the solution of (4.33) is to integrate (4.37) and sum the results.

5 Stability of numerical discretizations

Having shown that the coupled system (4.31) is well posed for $\mathbf{u} \in H^1$ and $z \in L^2$, we are in a good position to attempt to establish stability estimates for discrete approximations to its solutions. In other words, we wish to discretize (4.31) in such a way that it has at least one solution $(\mathbf{u}_h, p_h)$, and each solution $\mathbf{u}_h$ is uniformly bounded in H^1 and p_h in L^2, in terms of the data. However, a major complication is the divergence zero condition in (4.31). This is a well known difficulty with respect to numerical approximation of the Navier-Stokes equations (3.16), and thus it will not be easier here. Similarly, we limit our discussion to the 2-dimensional case as this is the only place we have any stability estimates for the continuum problem in weak norms suitable for numerical approximation.

The major new ingredient in the system (4.31) is the transport equation (4.33). Despite the difficulty in approximating (4.32), (4.33) is even harder.

Let us return to the original form of (4.33) as it appears as the third equation in (4.31), namely,

$$\nu z + \alpha \mathbf{u} \cdot \nabla z = \nu \operatorname{curl} \mathbf{u} . \tag{5.39}$$

Suppose that $\mathbf{u}_h = \mathbf{u}_h(z)$ and $p_h = p_h(z)$ denote finite element functions defined in appropriate spaces for solving (4.32) for a given $z \in L^2$. Let W_h denote a subspace of H^1 for simplicity in which we will seek approximations z_h to the solutions of (5.39).

We define a "master" variational problem for approximating (5.39) via a "centered" scheme as: find $z_h \in W_h$ such that

$$\nu\,(z_h, w) + \alpha\,(\mathbf{u}_h \cdot \nabla z_h, w) + \tfrac{1}{2}\alpha\,((\nabla \cdot\, \mathbf{u}_h) z_h, w) = \nu\,(\operatorname{curl} \mathbf{u}_h, w) \tag{5.40}$$

for all $w \in W_h$. Here, the forms (v, w) denote the $L^2(\Omega)$ inner-product. Note that we have allowed for the possibility that $\nabla \cdot \mathbf{u}_h$ is not exactly zero. (The schemes are called "centered" because the resulting "differencing" of the transport term corresponds to a centered, as opposed to "up-wind," difference.)

By combining (or iterating between) numerical approximations of (5.40) and (4.32) (substituting the result z_h of (5.40) for z in (4.32)) we obtain a coupled system for $\mathbf{u}_h$, p_h and z_h. Under suitable conditions which we now describe we can prove that the coupled system has solutions which converge to the system (4.31). We distinguish three cases for which we have determined results of varying degrees of completeness [18].

In the best case, suppose the approximation scheme for (4.32) has $\nabla \cdot \mathbf{u}_h$ identically zero [28] [29] [26] [2] [27] [19] [32] [33] [21]. In this case, the form multiplied by $\frac{1}{2}\alpha$ in (5.40) can be ignored in the formulation, and we can prove rates of convergence for $\mathbf{u} - \mathbf{u}_h$, $p - p_h$ and $z - z_h$ (under realistic smoothness assumptions on the data and domain). The bounds are quasi-optimal for $\mathbf{u} - \mathbf{u}_h$ and $p - p_h$ but suffer the usual degradation of order of approximation for $z - z_h$ that is common for centered schemes [18]. (For the notion of quasi-optimal approximation, see the article by Gunzburger in this volume.) For example, we can show for any Lipschitz polygon, and any p in the range $2 \leq p \leq 4$,

$$\begin{aligned} |\mathbf{u}_h - \mathbf{u}|_{W^{1,p}(\Omega)} \leq & |P_h(\mathbf{u}) - \mathbf{u}|_{W^{1,p}(\Omega)} \\ & + C_1 h^{2/p-1} |P_h(\mathbf{u}) - \mathbf{u}|_{H^1(\Omega)} + C_2 \|z - z_h\|_{L^2(\Omega)}, \end{aligned} \tag{5.41}$$

where $P_h(\mathbf{u})$ denotes the best approximation.

When the parameter $\frac{\alpha}{\nu}$ is not too large and the domain is convex, we can show that $z \in H^1(\Omega)$ and

$$\begin{aligned}\|z - z_h\|_{L^2(\Omega)} \leq 2\,\|z - R_h(z)\|_{L^2(\Omega)} &+ \sqrt{2}|\mathbf{u} - \mathbf{u}_h|_{H^1(\Omega)} \\ &+\frac{\alpha}{\nu}\Big(\|\mathbf{u} - \mathbf{u}_h\|_{L^\infty(\Omega)}|R_h(z)|_{H^1(\Omega)} \\ &\qquad + \|\mathbf{u}\|_{L^\infty(\Omega)}|z - R_h(z)|_{H^1(\Omega)}\Big),\end{aligned} \tag{5.42}$$

where $R_h(z)$ denotes the best approximation for z in W_h.

In the next best case, suppose the approximation scheme for (4.32) allows $\nabla \cdot \mathbf{u}_h$ to be non-zero but

$$\int_\Omega (\nabla \cdot \mathbf{u}_h) z_h w_h \, dx = 0 \quad \text{for any } z_h, w_h \in W_h. \tag{5.43}$$

Recall that the discrete divergence zero condition requires that $\nabla \cdot \mathbf{u}_h$ is orthogonal to anything in the pressure space Π_h. Thus if products $z_h w_h \in \Pi_h$ for any $z_h, w_h \in W_h$, then condition (5.43) holds. For example, suppose we take the Taylor-Hood approximation with Π_h consisting of continuous piecewise quadratics (and $\mathbf{u}_h$ a piecewise cubic), then we can take W_h to be piecewise linears. Again, the form multiplied by $\frac{1}{2}\alpha$ in (5.40) can be ignored in the formulation.

In the case that (5.43) holds, (5.42) remains valid while an interpolation error on the pressure has to be added to (5.41) in order to take into account the fact that $\nabla \cdot \mathbf{u}_h$ is not zero. Thus stability and rates of convergence are similar to those of the first case, except of course that the error for z is of the order of h since the elements of W_h are piecewise linears. The error could be h^2 if we pick W_h of degree two, but then the pressures must be of degree four and the velocities must be of degree five. This is not worth the expense, because in this case, one could use divergence-zero velocities. With divergence-zero velocities, we can achieve any order of accuracy by choosing W_h of high enough degree. Therefore, the compatibility condition (5.43) appears useful only if the elements of W_h are piecewise linears or piecewise constants.

Now suppose that we modify the transport approximation with a standard up-winding technique (streamline diffusion):

$$\begin{aligned}\nu\,(z_h, U(w)) + \alpha\,(\mathbf{u}_h \cdot \nabla z_h, U(w)) \\ +\tfrac{1}{2}\alpha\,((\nabla \cdot \mathbf{u}_h) z_h, w) = \nu\,(\operatorname{curl} \mathbf{u}_h, U(w)),\end{aligned} \tag{5.44}$$

where $U(w) = U(w; \delta, \mathbf{u}_h) := w + \delta \mathbf{u}_h \cdot \nabla w$ and δ is a small parameter linked with h [20]. Note that (5.40) can be viewed as a special case of (5.44) if we take $U(w) = w$. With the scheme (5.44), we can improve the rates of convergence for $\mathbf{u} - \mathbf{u}_h$, $p - p_h$ and $z - z_h$; in particular, we can obtain [18] a best-possible rate for z (i.e., the same as the best that can be obtained for the solution of a transport equation [20]).

Unfortunately, our estimates for the upwind scheme (5.44) involve constants proportional to $1/\alpha$ and would not be uniform as $\alpha \to 0$. This seems to be an artifact of the property that streamline diffusion schemes take serious account of the term $\alpha (\mathbf{u}_h \cdot \nabla z_h, w)$ whereas the centered schemes do not.

We have also been able to demonstrate convergence estimates in the above context (the Taylor-Hood approximation with Π_h consisting of continuous piecewise quadratics, the space for $\mathbf{u}_h$ consisting of piecewise cubics with two "bubble" functions, and W_h consisting of piecewise linears) when the "up-winding scheme" is the discontinuous Galerkin scheme [17].

Finally, suppose that we make no assumption about the approximation scheme for (4.32) with regard to $\nabla \cdot \mathbf{u}_h$. For example, suppose we take the standard Taylor-Hood approximation with Π_h consisting of continuous piecewise linears and $\mathbf{u}_h$ a piecewise quadratics. Take W_h to be continuous piecewise linears and consider the centered scheme (5.40). Then we can get stability and convergence results that are similar to those of the first case, but with larger constants.

6 Conclusions

We have examined a model for dispersive flow in two and three dimensions. We described several ways in which this model can be derived. We have shown that stability can be established in the 2-dimensional case. We described numerical schemes for approximating the equations and their corresponding stability and convergence properties.

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References

[1] Arnold, V. I. Sur la géométrie differentielle des groupes de Lie de dimension infinie et ses applicationa à l'hydrodynamique des fluids parfaits. *Ann. Inst. Fourier, Grenoble 16* (1966), 391–361.

[2] Bagheri, B., Scott, L. R., and Zhang, S. Implementing and using high–order finite element methods. *Finite Elements in Analysis & Design 16* (1994), 175–189.

[3] Benjamin, T. B., Bona, J. L., and Mahoney, J. J. Model equations for long waves in nonlinear dispersive systems. *Philos. Trans. Roy. Soc. London Ser. A 302* (1972), 47–78.

[4] Bona, J. L., Dougalis, V. A., Karakashian, O. A., and McKinney, W. R. Conservative, high-order numerical schemes for the generalized Korteweg-deVries equation. *Philos. Trans. Roy. Soc. London Ser. A 351* (1995), 107–164.

[5] Bona, J. L., Pritchard, W. G., and Scott, L. R. Solitary–wave interaction. *Physics of Fluids 23* (1980), 438–441.

[6] Bona, J. L., Pritchard, W. G., and Scott, L. R. An evaluation of a model equation for water waves. *Philos. Trans. Roy. Soc. London Ser. A 302* (1981), 457–510.

[7] Bona, J. L., Pritchard, W. G., and Scott, L. R. A comparison of solutions of two model equations for long waves. In *Fluid Dynamics in Astrophysics and Geophysics, N. R. Lebovitz, ed.,* (1983), vol. 20, Providence: Amer. Math. Soc., pp. 235–267.

[8] Bona, J. L., Pritchard, W. G., and Scott, L. R. Numerical schemes for a model for nonlinear dispersive waves. *J. Comp Phys. 60* (1985), 167–186.

[9] Camassa, R., and Holm, D. D. Thermodynamics, stability, and boundedness of fluids of complexity two and fluids of second grade. *Phys. Rev. Lett. 71* (1993), 1661–1664.

[10] Chen, S., Foias, C., Holm, D. D., Olson, E., Titi, E., and Wynne, S. The Camassa-Holm equations as a closure model for turbulent channel and pipe flow. *Phys. Rev. Lett. 81* (1998), 5338–5341.

[11] Cioranescu, D., Girault, V., Glowinski, R., and Scott, L. R. Some theoretical and numerical aspects of grade-two fluid models. *Research Notes in Math. 406* (1999), 99–110.

[12] Dunn, J. E., and Fosdick, R. L. Thermodynamics, stability, and boundedness of fluids of complexity two and fluids of second grade. *ARMA 56* (1974), 191–252.

[13] Dunn, J. E., and Rajagopal, K. R. Fluids of differential type: Critical review and thermodynamic analysis. *Int. J. Engng Sci. 33* (1995), 689–729.

[14] Ericksen, J. L., and Rivlin, R. S. Stress-deformation relations for isotropic materials. *ARMA 4* (1955), 323–425.

[15] Fosdick, R. L., and Rajagopal, K. R. Anomalous features in the model of second grade fluids. *ARMA 70* (1979), 145–152.

[16] Girault, V., and Scott, L. R. Analysis of a two-dimensional grade-two fluid model with a tangential boundary condition. *J. Math. Pures Appl. 78* (1999), 981–1011.

[17] Girault, V., and Scott, L. R. An upwind discretization of a steady grade-two fluid model in two-dimensions. Tech. rep., University of Chicago Computer Science, 2001.

[18] Girault, V., and Scott, L. R. Finite element discretizations of a two-dimensional grade-two fluid model. *MMAN accepted* (2002), 1–45.

[19] Ilin, A., , Bagheri, B., Metcalfe, R., and Scott, L. R. Error control and mesh optimization for high-order finite element approximation of incompressible viscous flow. *Comp. methods appl. Mechanics and Eng. 150* (1997), 313–325.

[20] Johnson, C., and Pitkäranta, J. An analysis of the discontinuous Galerkin method for a scalar hyperbolic equation. *Math. of Comp. 46* (1986), 1–26.

[21] Juarez, H., Scott, L. R., Metcalfe, R., and Bagheri, B. Direct simulation of freely rotating cylinders in viscous flows by high–order finite element methods. *Computers & Fluids in press* (1999), 1–36.

[22] Korteweg, D. J., and de Vries, G. On the change of form of long waves advancing in a rectangular channel, and on a new type of long stationary waves. *Philos. Mag. 39* (1895), 422–443.

[23] Marsden, J. E., and Shkoller, S. Global well-posedness for the Lagrangian averaged Navier-Stokes (LANS-α) equations on bounded domains. *Proc. Roy. Soc. London 359* (2001), 1449–1468.

[24] Ouazar, E. H. *Sur les Fluides de Second Grade.* Thèse de 3ème Cycle, Paris VI, 1981.

[25] Peregrine, D. H. Calculations of the development of an undular bore. *J. Fluid Mechanics 25* (1964), 321–.

[26] Scott, L. R., and Bagheri, B. Software environments for the parallel solution of partial differential equations. In *Computing Methods in Applied Sciences and Engineering IX, R. Glowinski and A. Lichnewsky, eds.* (1990), Philadelphia: SIAM, pp. 378–392.

[27] Scott, L. R., Ilin, A., Metcalfe, R., and Bagheri, B. Fast algorithms for solving high-order finite element equations for incompressible flow. In *Proceedings of the International Conference on Spectral and High Order Methods* (Houston, TX, 1995), University of Houston, pp. 221–232.

[28] Scott, L. R., and Vogelius, M. Conforming finite element methods for incompressible and nearly incompressible continua. In *Large Scale Computations in Fluid Mechanics, B. E. Engquist, et al., eds.* (1985), vol. 22 (Part 2), Providence: Amer. Math. Soc., pp. 221–244.

[29] Scott, L. R., and Vogelius, M. Norm estimates for a maximal right inverse of the divergence operator in spaces of piecewise polynomials. *M^2AN (formerly R.A.I.R.O. Analyse Numérique) 19* (1985), 111–143.

[30] Scott Russell, J. Report on waves. *Rep. Fourteenth Meeting of the British Assoc.* (1844), 311–.

[31] Taha, T., and Ablowitz, M. Analytical and numerical aspects of certain nonlinear evolution equations, part III: Numerical, Korteweg-deVries equation. *J. Comp. Phys. 55* (1984), 231–253.

[32] Tornberg, A.-K., Metcalfe, R., Scott, L. R., and Bagheri, B. A fluid particle simulation method. In *Computational Science for the 21st century* (1997), M.-O. Bristeau et al., Ed., John Wiley & Sons, New York, pp. 312–321.

[33] Tornberg, A.-K., Metcalfe, R., Scott, L. R., and Bagheri, B. A front-tracking method for simulating bubble motion in a viscous incompressible fluid using high–order finite element methods. In *1997 ASME Fluids Engineering Division Summer Meeting FEDSM97-3493* (1997).

[34] Whitham, G. B. *Linear and nonlinear waves.* Wiley, 1974.