Chapter Ten

Polynomial interpolation

Square brackets of the form [n.m] refer to display numbers in the book.

The expressions involving Newton divide differences can be confusing. The identity in [10.28] is an example. It may be useful to introduce the expression

$$f[\hat{x}_i] = f[x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n]$$

where the hat indicates what is left out. Then [10.28] becomes

$$f[x_0, \dots, x_n] = \frac{f[\hat{x}_i] - f[\hat{x}_j]}{x_j - x_i}.$$

Note the reversal of subscripts i and j between numerator and denominator. This is consistent with what we know:

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{f[\hat{x}_0] - f[\hat{x}_1]}{x_1 - x_0}.$$

There is a typo on the second line of [10.28]. The first term in the quotient should be $f[x_0, \dots$ and not $f[x_1, \dots$.

The behavior of Lagrange interpolation is mysterious. It is very sensitive to the distribution of higher derivatives of the function being interpolated as well as the distribution of the interpolation points. More precisely, we see from [10.25] and [10.27] that the quantities to focus on are $f^{(k)}/k!$ and the size of ω_k. This is most evident for Lagrange interpolation on uniformly spaced points. In Figure 10.1, we see the interpolant of the Runge function [10.37] together with the interpolant of a Gaussian. Superficially, the Runge function is the smoother of the two: they have the same value and curvature at the origin, but the Gaussian dives toward zero much more rapidly. Nevertheless, the interpolation error is significantly smaller for the Gaussian than for the Runge function once the number of interpolation points gets high enough.

The discrepancy has to do with the fact that the higher derivatives of the Gaussian grow more slowly than those of the Runge function. Indeed, Lagrange interpolation at a smaller number of (uniformly spaced) points, as shown in Figure 10.2 for 7 interpolation points, has a larger maximum error for the Gaussian (=0.381) than for the Runge function (=0.315). For larger numbers of points, Lagrange interpolation becomes sensitive to the higher derivatives which are not so easily visualized in a graph.

function interpolated	$n = 15$	17	19	21
Runge $1/(1+(3x)^2)$	0.937	1.32	1.89	2.73
Gaussian $e^{-(3x)^2}$	0.074	0.037	0.017	0.007

Table 10.1 Errors $\|f - L_n f\|_\infty$ for Lagrange interpolation L_n of a function f, where f is either the Runge function with $\gamma = 3$ or a Gaussian, as a function of the number n of equally spaced interpolation points.

As shown in Table 10.1, the interpolation increases for the Runge function but decreases for the Gaussian as the number of intepolation points is increased.

We can easily verify the relationship between the growth rates of the derivatives for the Runge function and a Gaussian. Let $f(x) = r_1(x) = 1/(1+x^2)$. Then the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}$$

diverges since f has a singularity in the complex plane at $\pm i$. On the other hand, $g(x) = e^{-x^2}$ is an entire function, so

$$\sum_{k=0}^{\infty} \frac{g^{(k)}(0) r^k}{k!} < \infty$$

for any finite value of r.

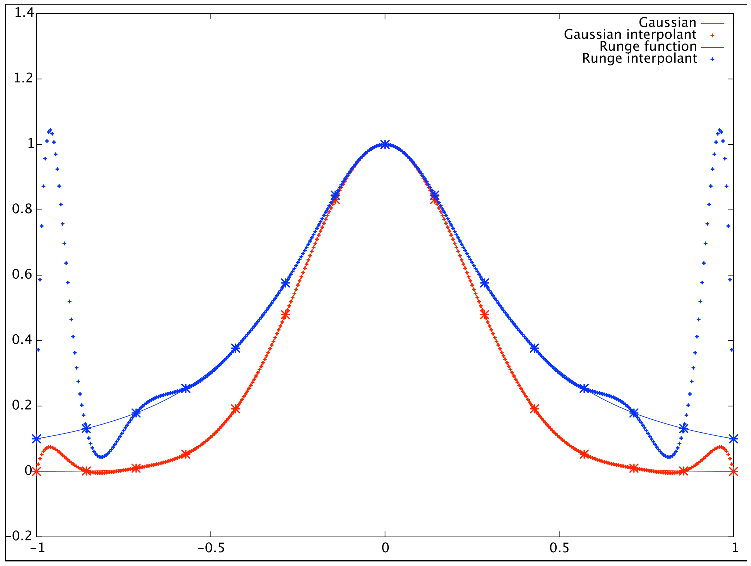

Figure 10.1 Interpolation of the Runge function and a Gaussian at 15 uniformly space points. The maximum interpolation error is approximately 0.937 for the Runge function and 0.0742 for the Gaussian.

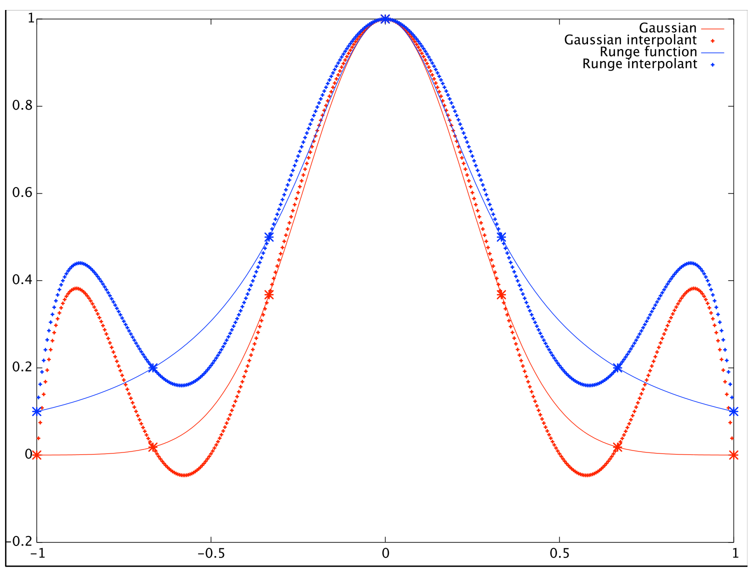

Figure 10.2 Interpolation of the Runge function and a Gaussian at 7 uniformly space points. The maximum interpolation error is approximately 0.315 for the Runge function and 0.381 for the Gaussian.

Chapter Eleven

Chebyshev and Hermite interpolation

Square brackets of the form [n.m] refer to display numbers in the book.

We can compare Lagrange interpolation for the Runge function and a Gaussian at Chebyshev points, as shown in Figure 11.1. For a modest number of interpolation points, the picture is not much different from what we saw with equally spaced points in Figure 10.2. The maximum interpolation error is reduced, but at the expense of a larger error in the middle of the interval. However, for larger numbers of points, the maximum error decreases for both functions, as indicated in Table 11.1. For example, Figure 11.2 shows very small error for 15 points.

Exercise 11.7
Equation [11.49] should instead read

$$\int_{-1}^{1} \omega_j(x)\omega_k(x)\frac{dx}{\sqrt{1-x^2}} = 0 \tag{11.1}$$

if $j \neq k$. This error is repeated on page 191 in the line just before the beginning of section 12.3.2.

function interpolated	$n=7$	11	15
Runge $1/(1+(3x)^2)$	0.096	0.027	0.0074
Gaussian $e^{-(3x)^2}$	0.110	0.014	0.0010

Table 11.1 Errors $\|f-L_n f\|_\infty$ for Lagrange interpolation L_n of a function f, where f is either the Runge function for $\gamma = 3$ or a Gaussian, as a function of number n of Chebysheb interpolation points.

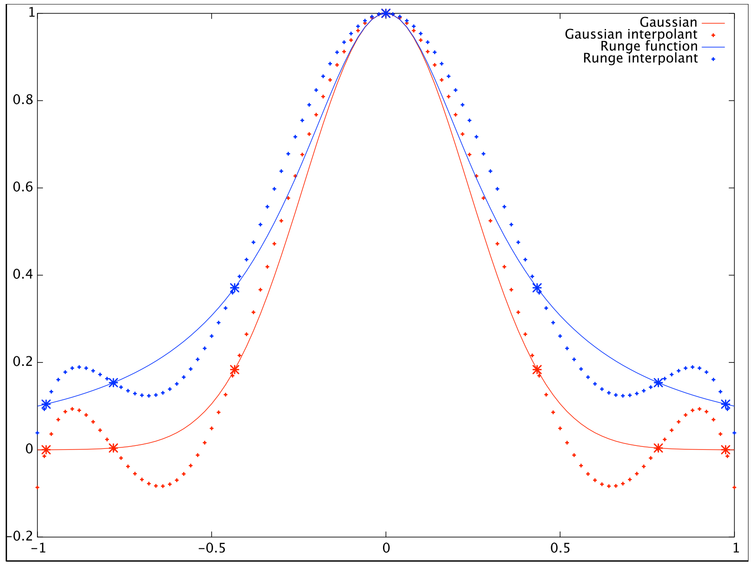

Figure 11.1 Interpolation of the Runge function and a Gaussian at 7 Chebyshev points. The maximum interpolation error is reduced at the expense of a larger error in the middle of the interval; compare Figure 10.2.

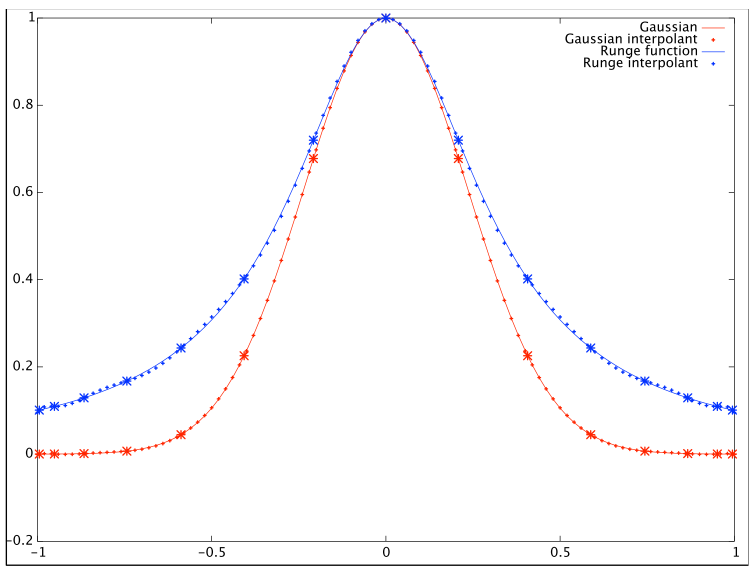

Figure 11.2 Interpolation of the Runge function and a Gaussian at 15 Chebyshev points. The maximum interpolation error is very small in the eye-ball norm for both functions; compare Figure 10.1.