1 Polynomial approximation, Chapter 5

1.1 Weierstrass and Bernstein, Chapter 5, Section 1

The Bernstein polynomial $B_n f$ can be used to prove Weierstrass' theorem. But the order of approximation is not optimal. Note that B_n is not a projection.

1.1.1 Homework

(35) pages 186-7, number 1
(36) Consider piecewise constant approximation on a uniform mesh of points i/n on $[0, 1]$. For a Lipshitz function, what is the best error estimate that you can give? Contrast this with the previous problem (35).

1.2 Lagrange interpolation, Chapter 5, Section 2

The existence of the Lagrange interpolant can be proved by constructing polynomials ϕ_i such that $\phi_i(x_j) = \delta_{ij}$. Then

$$Lf(x) = p(x) = \sum_{i=1}^{n} f(x_i)\phi_i \tag{1}$$

The operator L is a projection, since $p(x_i) = 0$ for $n + 1$ points implies that a polynomial p of degree n must be identically zero. In fact, this approach can be used to show (problem 27) the existence of the ϕ_i's.

The error in Lagrange interpolation satisfies

$$f(x) - Lf(x) = \frac{\prod_{i=0}^{n}(x - x_i)}{(n+1)!} f^{(n+1)}(\xi) = \frac{\omega_n(x)}{(n+1)!} f^{(n+1)}(\xi) \tag{2}$$

1.2.1 Homework

(37) page 193, number 2

1.2.2 Chebyshev interpolation, Chapter 5, Section 4.2

For equally spaced points, the size of ω_n is quite large at the ends of the interval; see Figs 1a and 1b on pages 268 and 269. Chebyshev points

$$x_j = \tfrac{1}{2}\left(1 + \cos\left(\frac{j\pi}{n+1}\right)\right) \tag{3}$$

lead to the optimal $\omega_n(x) = \cos(n \cos^{-1} x)$. The main point is that the Chebyshev points are distributed quadratically near the end points of the interval.

One can prove that

$$\|f - L_n^c f\|_{max} \le c(\log n) \inf_{p_n} \|f - p_n\|_{max} \quad (4)$$

Strangely, this is the best possible; there is no bounded linear projection onto polynomials. Note that the Bernstein operator is not a projection.

1.2.3 Homework

(38) page 229, number 1

1.2.4 Remainder term for Chebyshev and equidistant interpolation, Chapter 6, Section 3.1

This section computes precisely the differences in the error representation for Chebyshev and equidistant interpolation. It also has the Figs 1a and 1b on pages 268 and 269 which illustrate it.

1.2.5 Divergence of equidistant interpolation, Chapter 6, Section 3.4

This section estimates precisely the error for equidistant interpolation of $1/(1 + x^2)$ on $[-5, 5]$ and shows that the error diverges.

References