Notes for Math 211, Paris Center, winter 05

1 Polynomial approximation, Chapter 5

1.1 Weierstrass and Bernstein, Chapter 5, Section 1

The Bernstein polynomial $B_n f$ can be used to prove Weierstrass' theorem. But the order of approximation is not optimal. Note that B_n is not a projection. By (4c) on page 184, the Bernstein approximation to x^2 is

$$x^2 + \frac{x(1-x)}{n}$$

However, note that the Bernstein approximation is monotone in the sense that $B_n f \geq 0$ whenever $f \geq 0$.

1.1.1 Homework

(35) pages 186-7, number 1
(36) Consider piecewise constant approximation on a uniform mesh of points i/n on $[0,1]$. For a Lipshitz function, what is the best error estimate that you can give? Contrast this with the previous problem (35).

1.2 Lagrange interpolation, Chapter 5, Section 2

The existence of the Lagrange interpolant can be proved by constructing polynomials ϕ_i such that $\phi_i(x_j) = \delta_{ij}$. Then

$$Lf(x) = p(x) = \sum_{i=1}^{n} f(x_i)\phi_i \qquad (1)$$

The operator L is a projection, since $p(x_i) = 0$ for $n+1$ points implies that a polynomial p of degree n must be identically zero. In fact, this approach can be used to show (problem 27) the existence of the ϕ_i's.

The error in Lagrange interpolation satisfies

$$f(x) - Lf(x) = \frac{\prod_{i=0}^{n}(x - x_i)}{(n+1)!} f^{(n+1)}(\xi) = \frac{\omega_n(x)}{(n+1)!} f^{(n+1)}(\xi) \qquad (2)$$

1.2.1 Homework

(37) page 193, number 2

1.2.2 Chebyshev interpolation, Chapter 5, Section 4.2

For equally spaced points, the size of ω_n is quite large at the ends of the interval; see Figs 1a and 1b on pages 268 and 269. Chebyshev points

$$x_j = \tfrac{1}{2}\left(1 + \cos\left(\frac{j\pi}{n+1}\right)\right) \tag{3}$$

lead to the optimal $\omega_n(x) = \cos(n \cos^{-1} x)$. The main point is that the Chebyshev points are distributed quadratically near the end points of the interval.

One can prove that

$$\|f - L_n^c f\|_{max} \leq c(\log n) \inf_{p_n} \|f - p_n\|_{max} \tag{4}$$

Strangely, this is the best possible; there is no bounded linear projection onto polynomials. Note that the Bernstein operator is not a projection.

1.2.3 Homework

(38) page 229, number 1

1.2.4 Remainder term for Chebyshev and equidistant interpolation, Chapter 6, Section 3.1

This section computes precisely the differences in the error representation for Chebyshev and equidistant interpolation. It also has the Figs 1a and 1b on pages 268 and 269 which illustrate it.

1.2.5 Divergence of equidistant interpolation, Chapter 6, Section 3.4

This section estimates precisely the error for equidistant interpolation of $1/(1+x^2)$ on $[-5, 5]$ and shows that the error diverges.

The ratio of the error term for equally spaced points is larger than that of Chebyshev points by a factor of $(4/e)^n(\sqrt{2}/n)$ (see page 267).

1.3 Least squares, Chapter 5, Section 3

Another way to define approximations is by least squares. The basic polynomials are orthonormal:

$$(P_i, P_j) = \int_a^b P_i(x) P_j(x)\, w(x) dx = \delta_{ij} \tag{5}$$

where P_i is a polynomial of degree i of the form $P_i(x) = a_i x^i + q_i(x)$ where $a_i \neq 0$ and the degree of $q_i(x)$ is $i - 1$. This will be proved by induction.

This utilizes an inner-product structure on the linear space of square integrable functions. Then given any f define

$$L_n^S f = \sum_{i=0}^{n} (f, P_i) P_i. \tag{6}$$

Since the P_i's are linearly independent, the set $\{P_0, \ldots, P_n\}$ forms a basis for the space $\mathcal{P}_n$ of polynomials of degree n.

Define the norm associated with the inner-product by

$$\|f\|_2 = \sqrt{(f,f)} = \sqrt{\int_a^b f(x)^2 \, w(x) dx} \tag{7}$$

and we can see that

$$\|f - L_n^S f\|_2 = \min_{q \in \mathcal{P}_n} \|f - q\|_2 \tag{8}$$

Just note that

$$(f - L_n^S f, q) = 0 \tag{9}$$

for all q of degree n.

Suppose that $q \in \mathcal{P}_n$. Then $q = L_n^S q$ (L_n^S is a projection) because we must have $\|q - L_n^S q\| = 0$.

Moreover

$$\|f - L_n^S f\|_2 = 0 \tag{10}$$

if and only if $f \in \mathcal{P}_n$.

The polynomials can be defined by induction (called the Gram-Schmidt process, see section 3.1 of chapter 5): $P_0(x) = 1/(b-a)$ and

$$P_n = \frac{1}{\|x^n - L_{n-1}^S x^n\|_2} \left(x^n - L_{n-1}^S x^n \right) \tag{11}$$

Observe the following facts about P_n. First, we claim that the roots of P_n in the interval are all simple. For suppose that $P_n(x) = (x - x_1)^2 r(x)$. Then $P_n(x) r(x) = (x - x_1)^2 r(x)^2$ and we reach a contradiction that $(P_n, r) > 0$.

Secondly, all of the roots of P_n are in the interval $[a, b]$. To see this, enumerate all such roots with multiplicity as $x_0, \ldots, x_m$; let $q(x) = (x - x_0)(x - x_1) \cdots (x - x_m)$. Then $r = P_n q$ is of one sign in $[a, b]$. To see this, start them at the beginning with the same sign by adjusting the sign of q as necessary. At the next root, they both change sign since it is a simple root for both. So they stay of the same sign. This continues since all roots are simple. Thus $(P_n, q) > 0$. Since P_n is orthogonal to $\mathcal{P}_{n-1}$, we must have $m = n$.

1.3.1 Homework

(39) Prove (8) (hint: let q be arbitrary and consider the quadratic function of t defined by $\phi(t) := \|f - L_n^S f + tq\|_2^2$; use (9)).
(40) page 218, number 6

2 Differences, more interpolation, Chapter 6

We are skipping this, except for the parts already described relating to Lagrange interpolation.

- Chapter 6, Section 3.1: Remainder term for Chebyshev and equidistant interpolation,
- Chapter 6, Section 3.4: Divergence of equidistant interpolation

3 Numerical integration, Chapter 7

Quadrature is a way to approximate integrals.

3.1 Interpolatory quadrature, Chapter 7, section 1

The idea behind interpolatory quadrature is to define the approximate integral as the integral of an interpolant (or approximant):

$$Qf = \int_a^b Lf(x)\,dx = \sum_{i=0}^n f(x_i) \int_a^b \phi_i(x)\,dx \tag{12}$$

where L is one of the operators we constructed: Lagrange (you choose the points), Bernstein (unusual choice), piecewise-polynomial interpolant (this is called a composite rule). Least-squares approximation does not lead to such a quadrature rule because it is defined in terms of an integral. The error is easy to estimate:

$$Qf - \int_a^b f(x)\,dx = \int_a^b Lf(x) - f(x)\,dx \tag{13}$$

3.2 Newton-Cotes, Chapter 7, section 1.1

Here we have Lagrange interpolation with equally spaced points. It is of interest to note that the coefficients

$$\alpha_i := \int_a^b \phi_i(x)\,dx \tag{14}$$

are positive or not. For the open N-C, they are not. When the coefficients are negative, bad things can happen.

3.3 Gaussian quadrature, Chapter 7, section 3

Gaussian quadrature may be defined by taking the points x_i such that we get a formula exact for as high a degree as possible. Stated as a system of equations, it is highly nonlinear. With n values of x's and n values of α's, we might expect to integrate a polynomial of degree $2n-1$ exactly. Fortunately, if we take the x_i's to be the roots of the orthogonal polynomial P_n, all is well. First of all, we know the roots are in the interval in question and that they are distinct.

Suppose that $f \in \mathcal{P}_{2n-1}$. Then $f - L_n f$ vanishes at the roots of P_n, so we can write $f - L_n f = P_n q$ where $q \in \mathcal{P}_{n-1}$.

$$Qf - \int_a^b f(x)\,dx = \int_a^b Lf(x) - f(x)\,dx = \int_a^b P_n(x)q(x)\,dx = 0 \tag{15}$$

because P_n is orthogonal to $\mathcal{P}_{n-1}$.

Fortunately, the coefficients α_i are also positive. Let $f(x) = P_n(x)^2/(x-x_i)^2$. By (15), since the degree of f is $2n-2$,

$$\alpha_i f(x_i) = Qf = \int_a^b f(x)\,dx > 0 \tag{16}$$

since f is positive except at the x_i's. We also have $f(x_i) > 0$ since P_n has only a simple zero there.