Solving PDE’s with FEniCS

Wave equations

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There are two major classifications of PDEs, one for linear PDEs and one for nonlinearities. This is based on an algebraic trichotomy for second-order differential operators \( D \) in two dimensions: elliptic, parabolic, and hyperbolic.

This arises from the analogy with conic sections and their algebraic equations, as indicated in Table 1.

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<td>( x^2 + y^2 = 1 )</td>
<td>elliptic</td>
<td>Laplace</td>
<td>( u,xx + u,yy )</td>
<td>none</td>
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<td>( y = x^2 )</td>
<td>parabolic</td>
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<td>( u,t - u,yy )</td>
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<td>hyperbolic</td>
<td>wave</td>
<td>( u,tt - u,yy )</td>
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Table 1: Classification of linear PDEs. “formula” is name for a conic section.
Examples of elliptic equations are the Laplace/Poisson equations and the variants considered so far.

Heat equation is the prototypical parabolic equation.

In the classification in Table 1, this leaves hyperbolic equations.

Switching variables from $y$ to $x$, such an equation takes the form

\[ u_{tt} - u_{xx} = 0, \]  

(1)

in one space dimension.
More generally, what is called The wave equation is

\[ u_{tt} - c^2 \Delta u = 0, \quad (2) \]

in multiple space dimensions, where \( c \) is the wave speed and \( \Delta \) is the Laplacean.

The speed \( c \) may always be taken to be 1 in suitable coordinates.

For example, the speed of light in a vacuum is approximately 299792458 metres per second.
A meter is about 3.28084 feet.

So the speed of light in a vacuum is approximately $9.8357 \times 10^8$ feet per second.

Define **big foot** to be 1.0167 feet (about 12.2 inches), then speed of light is one big foot per nanosecond.

Approximately one giga-big-feet per second.

In either of these units, $c = 1$.

In one of them, time unit is small, and in other length unit is big.
Assume now that units are chosen so that $c = 1$.

Grace Hopper\(^1\) famously displayed in her talks a piece of copper wire whose length corresponded to the distance an electrical signal traveled in a nanosecond.

Her point was to emphasize the need for careful programming, but it also showed that $c = 1$ makes sense in a computer.

\(^1\)Grace Brewster Murray Hopper (1906–1992) was an early advocate for automating programming via the use of compilers to translate human-readable descriptions into machine code.
Other wave equations

There are other equations of higher order that do not fit the classification in Table 1, such as the dispersive Airy equation $u_t + u_{xxx} = 0$.

Using a more sophisticated classification [1], the Stokes equations are an elliptic system.

So we must consider Table 1 as just a starting point for understanding the differences between different PDEs.

But it does suggest a new equation to consider which is intimately related to wave motion.
The one-dimensional wave operator factors, so that (1) can be written

\[ 0 = u_{tt} - u_{xx} = \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) u \]

\[ = \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right) (u_t + u_x). \]  

(3)

In particular, \( u(x, t) = f(t - x) \) solves (3) for any function \( f \) of one variable, since \( u_t(x, t) = f'(t - x) \) and \( u_x(x, t) = -f'(t - x) \), and thus \( u_t + u_x = 0 \).

Thus solutions are constant on lines \( x(t) = t \).

Consist of just translating \( f \) to right at constant speed.
Waves go in both directions

Similarly, by re-ordering the factorization (3), we conclude that another family of solutions consist of translations to the left, without change of shape.

Such solutions are easy to visualize, and are markedly different from the behavior for the heat equation, or elliptic equations.

In particular, it appears at least formally that smoothness of $f$ does not matter, and that even discontinuous solutions would be allowed.
d’Alembert’s formula for solutions of wave equation:

\[
    u(x, t) = \frac{1}{2} f(t - x) + \frac{1}{2} f(t + x) + \frac{1}{2} \int_{x-t}^{x+t} g(s) \, ds,
\]

(4)

where \( g(x) = u_t(x, 0) \) and \( f(x) = u(x, 0) \).

The contributions of d’Alembert are memorialized in the name the wave operator

\[
    \Box u = u_{tt} - \Delta u,
\]

known as the d’Alembertian operator.
d’Alembert draws information from the light cone:

\[ u(x, t) = \frac{1}{2} f(t - x) + \frac{1}{2} f(t + x) + \frac{1}{2} \int_{x-t}^{x+t} g(s) \, ds, \]

Figure 1: The light cone for the wave equation.
Although corresponding relationship between data and solution is more complex in higher spatial dimensions, concept of light cone generalizes naturally.

Since waves tend to propagate without change of shape, the spatial domain $\Omega$ is often infinite.

In such cases, we truncate $\Omega$ for computational purposes.

The fact that information comes only from the light cone allows us to estimate the required size of the computational domain accurately.
One-way propagation

d’Alembert formula seems to imply waves always propagate in both directions.

But our splitting of the wave operator implied that there are solutions of the form $u_{\pm}(x, t) = f(x \pm t)$.

Suppose that $v(x, t) = f(x + t)$.

Then $v_t(x, 0) = f'(x)$.

So now consider the case where $f$ is given and $g(x) = f'(x)$ for all $x$. 
Either-way propagation

For \( g(x) = f'(x) \) for all \( x \), (4) gives

\[
2u(x, t) = f(x - t) + f(x + t) + \int_{x-t}^{x+t} f'(s) \, ds
\]

\[
= f(x - t) + f(x + t) + (f(x + t) - f(x - t)) = 2f(x + t).
\]

Thus \( u = v \) as expected. Similarly, if \( g(x) = f'( -x) \), then

\[
2u(x, t) = f(x - t) + f(x + t) + \int_{x-t}^{x+t} f'(s) \, ds
\]

\[
= f(x - t) + f(x + t) + (f(x - t) - f(x + t)) = 2f(x - t).
\]

So we get the expected one-way solutions by adjusting the initial impulse \( g \).
Since waves tend to propagate without change of shape, the spatial domain $\Omega$ is often infinite.

We truncate $\Omega$ for computational purposes.

Using standard algorithm, following heat equation, we obtain the variational expression

$$ (u_{tt}, v)_{L^2(\Omega)} + a(u, v) = 0 \quad \forall v \in H^1(\Omega), \quad (5) $$

where

$$ a(v, w) = \int_{\Omega} v'(x)w'(x) \, dx. \quad (6) $$

What is different is second-order time derivative.
We can approximate this, for example, via

$$u_{tt} \approx \frac{u^{n+1} - 2u^n + u^{n-1}}{\tau^2},$$

where $\tau > 0$ is the time step, and $u^j$ denotes an approximation to $u(j\tau)$. We can use this to define the variational formulation

$$(u^{n+1}, v)_{L^2(\Omega)} - 2(u^n, v)_{L^2(\Omega)} + (u^{n-1}, v)_{L^2(\Omega)} + \tau^2 a(u^{n+1}, v) = 0,$$

where $\tau$ is the time step. Thus we solve

$$(u^{n+1}, v)_{L^2(\Omega)} + \tau^2 a(u^{n+1}, v) = 2(u^n, v)_{L^2(\Omega)} - (u^{n-1}, v)_{L^2(\Omega)} \quad \forall v \in H^1(\Omega). \quad (7)$$
We immediately see that we can not solve the wave equation knowing just the initial conditions \( u(x, 0) \).

In addition, we need to know \( u_t(x, 0) \) and we can use this to get a good approximation to \( u^{-1} \).

Unfortunately, the time-stepping scheme (7) is only first-order accurate in time.
Another time-stepping scheme, advocated in [5], is

\[
(u^{n+1}, v)_{L^2(\Omega)} - 2(u^n, v)_{L^2(\Omega)} + (u^{n-1}, v)_{L^2(\Omega)}
= -\tau^2 \left( \theta a(u^{n+1}, v) + (1 - 2\theta) a(u^n, v) + \theta a(u^{n-1}, v) \right),
\]

where \( \theta \in [0, 1] \). Thus we solve

\[
(u^{n+1}, v)_{L^2(\Omega)} + \theta \tau^2 a(u^{n+1}, v) = 2(u^n, v)_{L^2(\Omega)} - (u^{n-1}, v)_{L^2(\Omega)}
- \tau^2 \left( (1 - 2\theta) a(u^n, v) + \theta a(u^{n-1}, v) \right), \quad \forall v \in H^1(\Omega).
\]

We can see the rationale for this approach by doing a Taylor expansion.
Taylor expansion

\[ u((n \pm 1)\tau) = u(n\tau) \pm \tau u_t(n\tau) + \frac{1}{2} \tau^2 u_{tt}(n\tau) \]
\[ \pm \frac{\tau^3}{6} u_{ttt}(n\tau) + O(\tau^4). \]  

(10)

Adding terms and dividing by \( \tau^2 \), and again making the correspondence \( u^j \approx u(j\tau) \), we find

\[ \frac{u^{n+1} - 2u^n + u^{n-1}}{\tau^2} = u_{tt}(n\tau) + O(\tau^2). \]

Similarly, multiplying (10) by \( \theta \) and adding gives

\[ \theta u((n + 1)\tau) + (1 - 2\theta)u(n\tau) + \theta u((n - 1)\tau) \]
\[ = u(n\tau) + \theta \tau^2 u_{tt}(n\tau) + O(\tau^4). \]  

(11)
Thus for any value of $\theta$ we get a second-order approximation:

$$
\frac{u^{n+1} - 2u^n + u^{n-1}}{\tau^2} + \theta u^{n+1} + (1 - 2\theta)u^n + \theta u^{n-1}
= u_{tt}(n\tau) + u(n\tau) + \mathcal{O}(\tau^2),
$$

and truncation error is decreasing in $\theta$.

If we take $\theta = 0$ we get an explicit scheme with time-step restrictions.

In [5], the choice $\theta = 1/4$ was advocated for stability reasons.
Figure 2: Error in $L^2(\Omega)$ as function of time for $\theta$ scheme with $\theta = 1/4$, $\Delta t = 0.05$, $L = 10$, $T = 5$, $M = 2000$ mesh points, using piecewise linears.
Start the simulation (in one space dimension) with

$$u_0(x) = e^{-x^2}$$

on domain $\Omega = [-L, L]$. Take $L$ as large as necessary.

Take $u_t(x, 0) = 0$. Then the exact solution follows from d’Alembert’s formula:

$$u(x, t) = \frac{1}{2}e^{-(t-x)^2} + \frac{1}{2}e^{-(t+x)^2}. \quad (13)$$

Error as a function of time is indicated in Figure 2.

Error increases rapidly initially but then decreases.
The conversion to higher space dimensions is notationally trivial.

We have written our bilinear form

\[ a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx \]

which makes sense in any number of dimensions.

Thus it is just a matter of defining the spatial domain to be multi-dimensional and waiting a bit longer for the answers to appear.
Consider solutions $U$ of wave equation (2) given by

$$U(x, t) = u(x)e^{ikt},$$

where $k \in \mathbb{R}$ and $i = \sqrt{-1}$. Then

$$\Delta U(x, t) = (\Delta u(x))e^{ikt} \quad \text{and} \quad U_{tt}(x, t) = -k^2 u(x)e^{ikt}.$$ 

Thus the wave equation

$$U_{tt}(x, t) - \Delta U(x, t) = f(x)e^{ikt}$$

is satisfied if

$$-k^2 u(x) - \Delta u(x) = f(x). \quad (14)$$
Equation (14) is known as **Helmholtz equation**.

Solving this (with boundary conditions) can be touchy since corresponding variational form not coercive.

If bothered by complex variable ansatz for \( U \), replace it with real counterparts, e.g.,

\[
U^1(x, t) = u(x) \cos(kt), \quad U^2(x, t) = u(x) \sin(kt),
\]

but the conclusion is the same:

\( U^i \)'s solve the wave equation if (14) holds.
Many types of wave behavior can be characterized as dispersive.

One example is water waves.

These are the waves that a child first sees.

The main characteristic of these waves is the way that they spread as time evolves.

Unlike the behavior we saw for The wave equation where waveforms just translate linearly in time as indicated in (4).
Dispersive waves

When the wave length $\lambda$ is long and the amplitude $a$ is small, compared to the water depth, approximate equations can be derived for surface wave behavior.

More precisely, define the Stokes parameter $S = a\lambda^2/d^3$, where $d$ is the (assumed uniform) depth of the water.

When $S = \mathcal{O}(1)$, then models for one-way propagation include the KdV equation [6, 3]

$$u_t + u_x + 2uu_x + u_{xxx} = 0,$$

where $u$ is the wave height.
Dispersive waves

The waves are assumed to be propagating in the $x$-direction, and constant in the $y$-direction.

We see that this equation is a nonlinear perturbation of the one-way wave equation $u_t + u_x = 0$.

Another, equivalent [3], model is

$$u_t + u_x + 2uu_x - u_{xxt} = 0.$$  \hfill (15)

The two equations related as perturbations of the basic wave equation $u_t + u_x = 0$.

Suggests the substitution $u_t \approx -u_x$, leads to

$$u_{xxt} \approx -u_{xxx}.$$
Simulations using equation (15) have been compared with laboratory experiments [2].

Solutions of (15) can be approximated numerically [4] by writing it in the form

\[
\left( I - \frac{\partial^2}{\partial x^2} \right) u_t = -u_x - 2uu_x = -(u + u^2)_x, \tag{16}
\]

where \( I \) denotes the identity operator.
The left-hand side is a familiar elliptic operator, so we can write a variational equation

\[ a(u_t, v) = -(u_x + 2uu_x, v)_{L^2(\Omega)} \]
\[ = -((1 + 2u)u_x, v)_{L^2(\Omega)}, \]  \hspace{1cm} (17)

where

\[ a(v, w) = \int_{\Omega} v(x)w(x) + v'(x)w'(x) \, dx. \]  \hspace{1cm} (18)

A variety of time-stepping schemes can be used to discretize (17).
One family of schemes that work well [4] is the Runge-Kutta schemes.

These schemes can be high order yet not require previous time values of the solution.

Often used as start-up schemes for other schemes that do utilize previous values, as do the $\theta$ schemes (8).

Simplest is **modified Euler** scheme: $\forall v \in V$

\[
a(\hat{u}^{n+1}, v) = a(u^n, v) - \Delta t \left( (1 + 2u^n)u_x^n, v \right)_{L^2(\Omega)} \\
a(u^{n+1}, v) = a(u^n, v) - \frac{1}{2}\Delta t \left( ((1 + 2u^n)u_x^n, v \right)_{L^2(\Omega)} + ((1 + 2\hat{u}^{n+1})\hat{u}_x^{n+1}, v)_{L^2(\Omega)} \right). \tag{19}
\]
The first step in (19) is called the **predictor step** and the second step in (19) is called the **corrector step**.

Figure presents example of this scheme with piecewise linear approximation in space.

The initial data \( u(x, 0) = e^{-x^2} \) evolves into a complex wave form that spreads out as it travels to the right, and even has a part that moves slowly to the left.

The wave speed is approximately \( 1 + 2u \), and in 25 time units the leading wave moves over 60 units to the right.

A linear wave (that is, if \( u \) were small) would have moved only 25 units to the right.
An example

A common measure of wave length for a complex wave shape is its width at half height.

For the function $f(x) = e^{-x^2}$, this is the point $x$ where $e^{-x^2} = 1/2$.

Thus $-x^2 = \log(1/2)$, so $x = \sqrt{\log 2} = 0.83 \ldots$.

Thus the initial wave length is less than 2, whereas the leading wave at $T = 25$ has a width that is about 7 units, over 4 times larger.

The dispersion causes the initial narrow wave to spread.
Despite the fact that there is both dispersion and nonlinearity in (16), it is possible to find special wave forms that propagate without change of shape.

These are called **solitary waves**.

For the equation (16), they take the form

\[
  u(x, t) = \frac{3}{2} A \text{sech}^2\left( \frac{1}{2} \sqrt{\frac{A}{A + 1}} (x - (1 + A)t) \right),
\]

where \( A > 0 \) can be any positive value. Recall that

\[
  \text{sech}(z) = \frac{2}{e^z + e^{-z}} = \frac{1}{\cosh(z)}.
\]


