Asymptotic notions of computability: minimal pairs and randomness

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https://cs.uchicago.edu/~royer/presentation.pdf

Intuition

"Definition"

A Turing machine M **solves** a problem P
if **for every** instance x of P,
M halts on x with the correct answer.

"Definition"

A Turing machine M **asymptotically solves** a problem P
if **for almost every** instance x of P,
M halts on x with the correct answer.

Density Definition

Definition

A subset A of $\mathbb{N}$ is **dense** if

$$\lim_{n\to\infty} \frac{A \cap [0,n)}{n} = 1$$

and **sparse** if the limit is 0.

Definition

A subset B of $\{0,1\}^*$ is **dense** if

$$\lim_{n\to\infty} \frac{|\{\sigma \in B : |\sigma| = n\}|}{2^n} = 1$$

and **sparse** if the limit is 0.

Coarse and Generic computability

Definition

A set A is **coarsely computable**
if there exists a Turing machine M such that
$M(x)\downarrow$ for all x and the set

$$\{x \mid M(x) = A(x)\}$$

is dense.

Definition

A set A is **generically computable**
if there exists a Turing machine M such that
$M(x)\downarrow$ implies $M(x) = A(x)$ and the set

$$\{x \mid M(x)\downarrow\}$$

is dense.

Examples

Example

Every computable set is both coarsely and generically computable.

Example

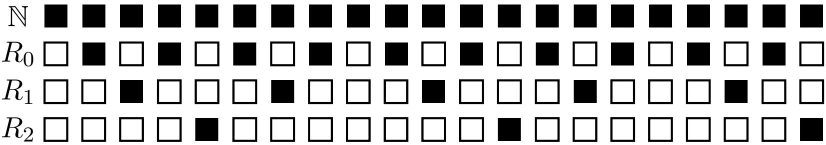
The set

$$A = \{2^n \mid n \in \text{Halting}\}$$

is not computable, but it is both coarsely and generically computable.

Examples

$$R_e = \{2^e(2l+1) \mid l \in \mathbb{N}\}$$



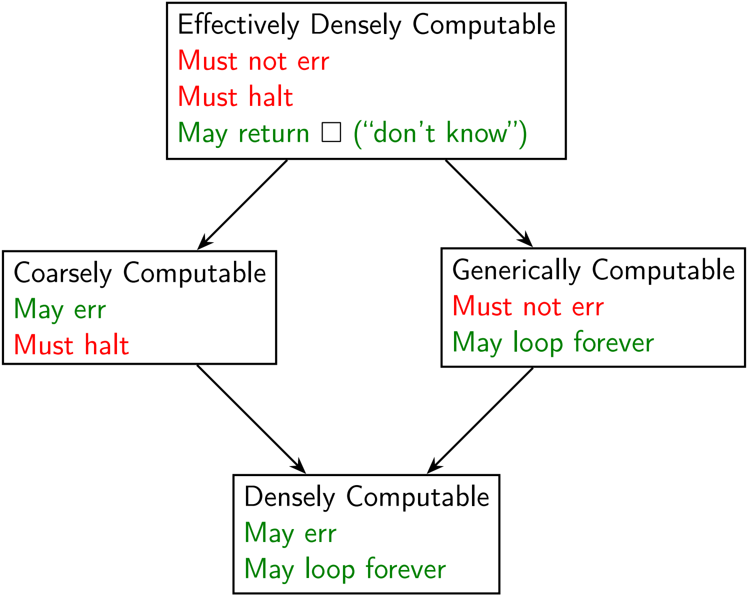
Example

The set

$$B = \bigcup_{n \in \text{Halting}} R_n$$

is coarsely computable, but not generically computable.

Four Horsemen of Asymptotic Computability



Minimal Pairs in the Turing Degrees

Definition

A **minimal pair for the Turing degrees**
is a pair of sets A, B such that
neither A nor B are computable,
but if C is computable relative to both A and B,
then C is computable.

Theorem (1950's)

There exists a minimal pair for the Turing degrees.

Minimal Pairs for Relative Coarse Computability

Definition

A **minimal pair for relative coarse computability**
is a pair of sets A, B such that
neither A nor B are coarsely computable,
but if C is coarsely computable relative to both A and B,
then C is coarsely computable.

Theorem (Hirschfeldt, Jockusch, Kuyper, Schupp, 2016)

There exists a minimal pair for relative coarse computability.

Minimal Pairs for Relative Asymptotic Computability

Theorem (Astor, Hirschfeldt, Jockusch, 2019)

There exists a minimal pair for relative dense computability.

Theorem (Hirschfeldt, Jockusch, Kuyper, Schupp, 2016)

There exists a minimal pair for relative coarse computability.

Theorem (Igusa, 2013)

*There are **no** minimal pairs for relative generic computability.*

Theorem (Igusa, 2013)

*There are **no** minimal pairs for relative effectively dense computability.*

Coarse Reducibility

Definition (rephrased)

A set A is **coarsely computable relative to** B
if there exists a Turing functional Φ such that
the set Φ^B is a coarse description of A.

- Note the assymetry! The input is the set B itself, but the output is a coarse approximation to A

Definition

A set A is **coarsely reducible** to B
if there exists a Turing functional Φ such that
for all coarse descriptions C of B,
the set Φ^C is a coarse description of A.

Minimal Pairs for Asymptotic Reducibilities

Theorem (Astor, Hirschfeldt, Jockusch, 2019)

There exists a minimal pair for dense reducibilities.
In fact, there are measure-1 many such pairs.

Theorem (Hirschfeldt, Jockusch, Kuyper, Schupp, 2016)

There exists a minimal pair for coarse reducibilities.
In fact, there are measure-1 many such pairs.

Theorem (Hirschfeldt, 2020)

There exists a minimal pair for generic reducibility.

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Theorem (Hirschfeldt, 2020)

There exists a minimal pair for generic reducibility.

Open Problem

Are there minimal pairs for effective dense reducibility?

Very Few Minimal Pairs for Generic Reducibility

Theorem (R)

There are only measure-0 many minimal pairs for generic reducibility.

Proof sketch.

Igusa constructs a pair of Turing functionals Φ, Ψ
and two countable lists $\{X_e\}_{e\in\mathbb{N}}, \{Y_e\}_{e\in\mathbb{N}}$
such that, if $A \neq X_e$ and $B \neq Y_e$ for all e,
then $\Phi^A \cup \Psi^B$ is not generically computable,
but is generically computable relative to both A and B.

Very Few Minimal Pairs for Generic Reducibility

Theorem (R)

There are only measure-0 many minimal pairs for generic reducibility.

Proof sketch.

We construct a pair of Turing functionals Φ, Ψ
and two countable lists $\{X_e\}_{e \in \mathbb{N}}, \{Y_e\}_{e \in \mathbb{N}}$
such that, if $A \triangle X_e$ and $B \triangle Y_e$ are not sparse for any e,
then $\Phi^A \cup \Psi^B$ is not generically computable,
but is **generically reducible** to both A and B. □

1-Randomness

Definition

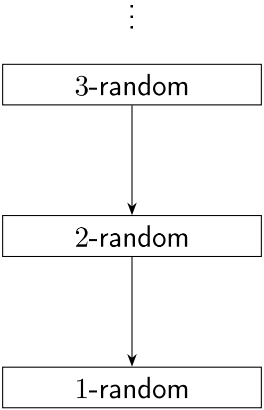
A **Martin-Löf test** is a uniform sequence $U_0 \supseteq U_1 \supseteq \cdots$ of Σ^0_1 classes such that $\mu(U_i) < 2^{-i}$.
A set A passes the test if $A \notin \bigcap_i U_i$.
A set is 1-random if it passes all Martin-Löf tests.

n-Randomness

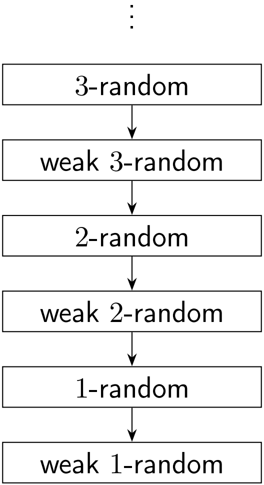
Definition

A Σ^0_n **Martin-Löf test** is a uniform sequence $U_0 \supseteq U_1 \supseteq \cdots$ of Σ^0_n classes such that $\mu(U_i) < 2^{-i}$.
A set A passes the test if $A \notin \bigcap_i U_i$.
A set is n-random if it passes all Σ^0_n Martin-Löf tests.

n-Randomness



n-Randomness and Weak Randomness



Randomness and Minimal Pairs

Theorem (Astor, Hirschfeldt, Jockusch, 2019)

If A and B are relatively weakly 4-random, then they form a minimal pair for dense reducibility.

Theorem (Hirschfeldt, Jockusch, Kuyper, Schupp, 2016)

If A and B are relatively weakly 3-random, then they form a minimal pair for coarse reducibility.

Theorem (R)

*If A and B are weakly 2-random, then they **do not** form a minimal pair for generic reducibility.*

Theorem (R)

*If A and B are weakly 2-random, then they **do not** form a minimal pair for effectively dense reducibility.*

Randomness and Minimal Pairs

Theorem (R)

If A and B are weakly 2-random,
*then they **do not** form a minimal pair for generic reducibility.*

Proof sketch.

Recall: there are $\Phi, \Psi, \{X_e\}_{e\in\mathbb{N}}, \{Y_e\}_{e\in\mathbb{N}}$
that if $A \triangle X_e$ and $B \triangle Y_e$ are not sparse for all e,
then $\Phi^A \cup \Psi^B$ witnesses that A, B are not a minimal pair.
Argue that X_e, Y_e are $\emptyset'$-computable,
thus A, B are weakly 1-random relative to all X_e, Y_e,
and use lemma below. □

Lemma

If A is weakly 1-random relative to X,
then $A \triangle X$ is not sparse.

Randomness and Minimal Pairs

	Relative Computability			
	Dense	Coarse	Generic	Eff. dense
Min. pairs?	Yes	Yes	No	No
How many?	Measure-1	Measure-1	-	-
Randomness?	Weak 4	Weak 3	-	-
	Reducibility			
	Dense	Coarse	Generic	Eff. dense
Min. pairs?	Yes	Yes	Yes	?
How many?	Measure-1	Measure-1	Measure-0	$\leq$ Measure-0
Randomness?	Weak 4	Weak 3	Weak 2	Weak 2

Related Open Problems

Open Problem

Can we show that if A and B are 1-random,
then they do not form a minimal pair for generic reducibility?

Open Problem

For each of the asymptotic reducibilities,
is every function equivalent to the indicator function of a set?

Open Problem

Which reducibilities imply each other?

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Partial Results on Sets vs. Functions

Theorem (R)

If $\log f(n) \leq n^{O(1)}$ *then* f *is equivalent to a set under all four reducibilities.*

Definition

A **simple encoding** is a function $f : \mathbb{N} \rightarrow 2^{\mathbb{N}}$ such that if $x \neq y$ then $E(x) \cap E(y) = \emptyset$.
For $f : \mathbb{N} \rightarrow \mathbb{N}$, define E_f by

$$E_f = \bigcup_{n \in \operatorname{dom} f} E(\langle n, f(n) \rangle).$$

Theorem (R)

If E *is a simple encoding, then there exists an* f *such that* f *and* E_f *are not equivalent under any of the four reducibilities.*