**MORE INDUCTION**

\[ H_n = \sum_{i=1}^{n} \frac{1}{i} \]  

**Harmonic Numbers**

**Claim**  
\[ H_{2^n} \geq 1 + \frac{n}{2} \]

**Corollary**  
\[ \lim_{n \to \infty} H_n = +\infty \]  
[Def. of \( \lim \)  \( \forall k \exists j \text{ s.t. } H_j > k \)]

**proof**  
**Basis Step**  
\[ n = 0 \]  
\[ H_2^0 = H_1 = 1 \]  
\[ 1 + \frac{0}{2} = 1 \]

\[ \text{LHS} = \text{RHS} \]  
**done**

**INDUCTION STEP**

\( \Phi(k) : H_{2^k} \geq 1 + \frac{k}{2} \)

Need to prove \( \Phi(k+1) \)  
\[ H_{2^{k+1}} \geq 1 + \frac{k}{2} \]

\[ H_{2^{k+1}} = \sum_{i=1}^{2^{k+1}} \frac{1}{i} = \sum_{i=1}^{2^k} \frac{1}{i} + \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i} \geq 1 + \frac{k}{2} + \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i} \]

\[ \text{I.H.} \]

\[ \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i} = \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} + \ldots + \frac{1}{2^{k+1}} \]  
\[ 2^k \text{ terms} \]

\[ \geq \frac{1}{2} \]  
(>)

So \( \Phi(k+1) \) holds. So proven by induction. \( \square \)

**Claim**  
\[ \forall n \geq 57 \left[ 7^{n+2} + 8^{2n+2} \right] \]

(a bit tricky) inductive step. Assuming for \( k \), \( \Phi(k+1) = 7^{k+1+2} + 8^{2(k+1)+2} \) divisible by 57

Since  
\[ 7^{k+1+2} = 7^{k+2} \cdot 7 \]
\[ 8^{2k+2+2} = 64 \cdot 8^{2k+2} \]

\[ 7^{k+1+2} + 8^{2(k+1)+2} = 7 \left[ 7^{k+2} + 8^{2k+2} \right] + 57 \cdot 8^{2k+2} \]

\[ \text{divisible by 57} \]
Equivalent: if $S \subseteq \mathbb{N}$ then $S$ has a smallest element

$$\min \{x | x \in S\}$$

'Well-ordering property' [used in proof of correctness of division algorithm]

Suppose induction false: i.e., $\varphi(0)$

$$(\forall k)[\varphi(k) \rightarrow \varphi(k+1)]$$

but $\neg (\forall n \varphi(n))$

Then $\exists m, \neg \varphi(m)$

Let $S = \{ k | \neg \varphi(k) \}$ and let $x = \min(S)$

$x \neq 0$

$$(x-1) \in \mathbb{N} \wedge \neg \varphi((x-1))$$

but by induction $\varphi((x-1)) \rightarrow \varphi(x)$

Other direction - exercise

Round-Robin Tournament $\{ p_1, \ldots, p_n \}$ $p_i \ldots p_n$ players

$\forall i \neq j, p_i$ plays $p_j$, no draws. $p_i$ wins $p_j$ $p_j$ wins $p_i$

[needed?]

Claim: If there is a cycle, there is one of length 3.

Suppose false: $\exists$ cycle, $\neg \exists$ cycle w/ 3 players.

Consider a shortest cycle $\{ q_1, \ldots, q_n \}$ [Why?]

Consider $q_1, q_2, q_3$

What is the result of the game between $q_1$ and $q_3$?

$q_1 \rightarrow q_3$ forms a $\Delta$; Contradiction

But then $q_3 \rightarrow q_1$ and

$q_1, q_3, q_4 \ldots q_n, q_1$ is a shorter cycle.

Contradiction