Generating Functions

Sequence \( a_0, \ldots, a_k, \ldots \) \( \rightarrow \) \( G(x) = \sum_{k=0}^{\infty} a_k x^k \)

Generating function of the sequence

Formal series - do not worry about convergence

For finite seq. \( a_0, \ldots, a_k \) \( \rightarrow \) \( a_0 - a_k 0 0 0 \ldots \) use definition

\[
1 + 1 + 1 + \ldots = 1 + x + x^2 + x^3 = \frac{x^4 - 1}{x - 1} \quad x \neq 1
\]

\( G(x) \rightarrow a_0, \ldots, a_k \ldots \)

\[
\frac{1}{1 - ax} = 1 + ax + a^2 x^2 + a^3 x^3 + \ldots \quad (|ax| < 1 \text{ converges } |x| < \frac{1}{|a|})
\]

\[
(1 + x)^\eta = \binom{\eta}{0} + \binom{\eta}{1} + \binom{\eta}{2} + \ldots
\]

Operations

\[
F(x) = \sum_{k=0}^{\infty} a_k x^k \quad G(x) = \sum_{k=0}^{\infty} b_k x^k
\]

\[
+ \sum_{k=0}^{\infty} (a_k + b_k) x^k
\]

\[
\times \sum_{k=0}^{\infty} \sum_{i=0}^{k} (a_i b_{k-i}) x^k
\]

\[
\frac{d}{dx} F(x) = \sum_{k=0}^{\infty} (k+1) a_k x^k
\]

\[
\left[ a_0 + a_1 x + a_2 x^2 + \ldots + a_c x^c + a_{c+1} x^{c+1} \right]
\]

Formal derivative of each term

\[
0 + a_1 + 2a_2 x + \ldots + k a_k x^{k-1} + (k+1) a_{k+1} x^k + \ldots
\]

When the series converge, these are the rules of calculus...

Application: \( \frac{1}{1-x} \) is the q.f. \( \frac{d}{dx} (1 + x + x^2 + \ldots) = F(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \)

\[
\frac{d}{dx} \frac{1}{(1-x)^2} = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots + k x^{k-1} = \sum_{k=1}^{\infty} k x^{k-1}
\]

So \( \sum_{k=0}^{\infty} k 2^{k-1} = \frac{1}{(1-1/2)^2} = 4 \)
Extended binomial coefficients:
\[
\binom{u}{m} = \begin{cases} 
\frac{u(u-1)\ldots(u-m+1)}{m!} & \text{if } m > 0 \\
1 & \text{if } m = 0
\end{cases}
\]

One can check that for \( u \in \mathbb{R} \), \( u \neq 0 \), \( u = -n \)
\[
\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}
\]

since
\[
\frac{(-n)(-n-1)(-n-2)\ldots(-n-k+1)}{(n-1)!} = \frac{(-1)^k n(n+1)\ldots(n+k-1)}{k!}
\]
\[
= \frac{(n-1)!}{(n-1)!} \frac{(-1)^k n(n+1)\ldots(n+k-1)}{k!} = (-1)^k \frac{(n+k-1)!}{k!(n-1)!}
\]

(Extended) Binomial Theorem
\[
(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k
\]

(this is the one attributed to Newton)

will not prove

Uses of Generating Functions

Can use the technique in the proof of the (natural number) Binomial Theorem to
express the number of solutions of counting problem as the coefficient of some \( x^k \)
in a generating function.

See Examples 11, 12, 13, 14, and 15 in Rosen.

Solve recurrence relations:

Note that if \( G(x) = \sum_{n=0}^{\infty} a_n x^n \) is the generating function of \( a_n \),
\[
\log G(x) = \sum_{n=0}^{\infty} a_n n x^n
\]
\[
\Rightarrow G(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n+1}
\]
\[
a_n x^n = \frac{a_n x^n}{n+1} \]

So we can solve \( a_n = a_{n-1} \)
by noting that  
\[ G(x) = \sum_{k=0}^{\infty} a_k x^k = \frac{a_0}{1 - 2x} = a_0 \prod_{k=0}^{1} \left( x - a_k \right) \]

So we can solve for \( G(x) \)

\[ G(x) = \frac{a_0}{1 - 2x} \]

\[ a_n = 2^{n+1} \]

Such manipulations sometimes need to transform expressions of the form \[ \frac{b}{(x-r)(x-s)} \]
into ones of the form \[ \frac{A}{x-r} + \frac{B}{x-s} \] (because we know the generating functions of \( \frac{1}{x-s} \)).

The trick (from Calculus, where it was used to reduce the computation)

\[ \frac{d}{dx} \left( \frac{b}{(x-r)(x-s)} \right) \]

is to try to have \( \frac{A}{x-r} + \frac{B}{x-s} = \frac{\frac{b}{(x-r)(x-s)}}{\frac{1}{x-r}(x-s)} \)

we get \( \frac{A(x-s) + B(x-r)}{(x-r)(x-s)} = \frac{\frac{b}{(x-r)(x-s)}}{\frac{1}{x-r}(x-s)} \)

we get \( A(x-s) + B(x-r) = \alpha \)

\( (A+B)x - (A+B) = \alpha \)

so we must have \( \begin{cases} A + B = 0 \\ A + B = 0 \end{cases} \)

which is a contradiction.

\[ \frac{b}{(x-r)(x-s)} \]

we need a completely unrelated expression...