Please put your name on every page of the test.

This is a closed book, closed neighbor test.

The test has 6 questions for a total of 55 points, and an Extra Credit question worth 15 points. Only work on the Extra Credit if you have used all the time needed on the normal problems.

Please be concise and precise.

In a proof by induction, be sure to state the Induction Hypothesis precisely.

Good luck!
1. (5 points) Compute \((5^{1001} \mod 11)^2 \mod 11\)

Use Fermat’s little Theorem; for \(p\) prime \(a^{p-1} \equiv 1 \pmod{p}\) for all \(a \in \mathbb{Z}\).

11 is prime, so \(p-1 = 10\) gives us \(5^{1001} \equiv 5^1 \pmod{11}\) and

\[5^2 = 25 \quad 25 \equiv 3 \pmod{11}\]

Some people claimed from \(1000 = 10^3\) that \(5^{1000} = (5^{10^3})^3\)!

Some used the more cumbersome \(5^{11}\) repeatedly.

Some tried to use fast exponentiation, computing \(5^{10^3}\) — usually this yielded computational mistakes.

\(*\) \(\gcd(a, p) = 1\) would be sufficient
2. (15 points) Consider three positive integers, \( a, b \) and \( c \). We are given that \( a | bc \) (\( bc \) is a multiple of \( a \)) and that \( a \) does not divide \( b \). For each of the two statements below, either prove that they are correct, or show that they are not correct (you may provide a counterexample.)

1. Assume that \( \gcd(b, c) = 1 \). Then \( a | c \).

2. Assume \( c > b \). Then \( a | c \).

3. Assume \( a \) is a prime. Then \( a | c \).

\[ a = 6, \ b = 4, \ c = 9 \] is a counterexample to both 1) and 3).

3. is true.

The proof from Bézout's theorem: if \( a \) is prime and \( a | b \) then \( \gcd(a, b) = 1 \) and \( \exists s, t: 1 = as + bt \)

Multiplying by \( c \) \( c = acs + bct \)

\( c \equiv acs + bct \pmod{a} \)

\( c \equiv 0 \pmod{a} \) since \( a | bc \)

\( \) or \( bc = xa \) for some nonzero \( a \)

so \( a | c \)

Proof using prime decomposition (Fundamental Theorem of Arithmetic)
Consider the primes in the prime decomposition of \( bc \). \( a \) must be in this set, since \( a \) is prime and \( a | bc \), \( a \) is not among the primes in the decomposition of \( b \) (if it were, \( a | b \)). The primes in the decomposition of \( bc \) are the union of the ones in the decomposition of \( b \), and \( c \). So \( a \) must be in the prime decomposition of \( c \).
3. (10 points) Prove that for every integer \( n \geq 1 \)
\[
1 \times 2 + 2 \times 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}
\]

The \( \text{IH} \): \( \varphi(n) \)
\[
1 \cdot 2 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}
\]

Want to prove \( \forall n \varphi(n) \)

Proof by induction.

\text{basis } \quad n = 1
\[
\text{LHS } = 1 \cdot 2 = 2 = \frac{1 \cdot 2 \cdot 3}{3} = \text{RHS}
\]

\( \varphi(1) \) is true,

\text{induction step } \quad \text{Assuming } \varphi(n), \text{ prove } \varphi(n+1)

\text{LHS for } n+1
\[
1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) + (n+1)(n+2) =
\]
\[
= \frac{n(n+1)(n+2)}{3} + (n+1)(n+2) \quad \text{(using IH)}
\]
\[
= \frac{n(n+1)(n+2)}{3} + \frac{3(n+1)(n+2)}{3} = \frac{(n+1)(n+2)(n+3)}{3} = \text{RHS}
\]

So \( \varphi(n) \Rightarrow \varphi(n+1) \), for general \( n \).

\( \square \)

You must state IH.

Proofs that manipulate both sides are syntactically suspect.
4. (5 points) Given a group of 8 women and a group of 6 men, how many different ways are there to select a team of 3 players that has at least one player of each sex? [You do not have to do the arithmetic]

Several correct strategies:

a) You can select 1 or 2 women for the team.

Can select 1 woman \( \binom{8}{1} \) ways. For each selection there are \( \binom{6}{2} \) groups of 2 men, \( \binom{8}{1}\binom{6}{2} \) total.

Can select 2 women \( \binom{8}{2} \) ways, giving \( \binom{8}{2}\binom{6}{1} \) teams.

\( \binom{8}{1}\binom{6}{2} + \binom{8}{2}\binom{6}{1} \) is the answer.

b) You can have \( \binom{8+6}{3} \) teams. All-male teams - \( \binom{6}{3} \) of them - must be excluded, as should the \( \binom{8}{3} \) all-female teams.

\( \binom{14}{3} - \binom{8}{3} - \binom{6}{3} \)

c) We can select 1 female - \( \binom{8}{1} \) ways, 1 male - \( \binom{6}{1} \) ways, and another player - \( \binom{12}{1} \) ways (we already used 2 of the 14 players).

\( \binom{8}{1}\binom{6}{1}\binom{12}{1} = 8 \cdot 6 \cdot 12 \) counts every team twice - for example, the team \( F_1 F_2 M \) is counted as 'choose \( F_1 \) from the 8 females, choose \( M \) from among the 6 males, then choose \( F_2 \) from the remaining 12 players,' and also as 'choose \( F_2 \) from the 8 females... etc.'

So we divide by 2 to get the correct result \( \frac{8 \cdot 6 \cdot 12}{2} \).
5. (10 points) Prove:
\[ \sum_{k=0}^{n} 3^k \binom{n}{k} = 4^n \]

Use the Binomial Theorem
\[ (a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \]
with \( a = 3 \) and \( b = 1 \)
\[ 4^n = (3+1)^n = \sum_{k=0}^{n} \binom{n}{k} 3^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k} 3^k \]
(since \( 1^k = 1 \) for any \( k \))

It is possible to give a proof using induction and
Pascal's identity \( \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \) as well as careful algebra and summations. It is very difficult to do it with the time pressure of an exam.
6. (10 points) Say \textit{precisely} what is wrong with the proof below:

\textbf{Theorem}(!) For every nonnegative integer \(n\), \(5n = 0\).

\textbf{Proof by induction.} The Inductive Hypothesis is \(5k = 0\)

\textbf{(Base Case)} For \(k = 0\), \(5k = 5 \times 0 = 0\)

\textbf{(Inductive step.)} We assume the inductive hypothesis is true for every \(k < n\) and we will prove it for \(n\).

\begin{itemize}
  \item We can write \(n = i + j\), where \(i\) and \(j\) are both nonnegative integers, and both are less than \(n\)
  \item Now we can apply the induction hypothesis to \(i\) and \(j\): this shows \(5i = 0\) and \(5j = 0\)
  \item So \(5n = 5(i + j) = 5i + 5j = 0 + 0 = 0\)
  \item This proves the induction step.
\end{itemize}

\textbf{Note}: your answer must be very precise and concrete. Do not say “the conclusion cannot be right”. This is clear, but it is not what the question is.

\textbf{The problem is the statement } *

\begin{itemize}
  \item If \(n = 1\) if \(i < n\), \(j < n\) \(i = j = 0\) and \(i + j = 0 \neq n\)
  \item So that statement is not true.
\end{itemize}

\textbf{There were many wrong answers. They mostly pointed out that the conclusion of the theorem is false, or that } \(5 \cdot 1 \neq 0\). \textbf{This does not say what is wrong with the proof: it refers to the conclusion.}

\textbf{There were claims essentially saying \textquote{this is not the right scheme for induction}. Those are wrong claims. Everything in the proof is fine—except for the statement with *}. 
7.[Extra Credit] (15 points) Let \( A \) be a set of 10 distinct positive integers between 1 and 50 (i.e. if \( x \in A, 1 \leq x \leq 50 \)). Consider the collection \( S \) of all 5-subsets of \( A \) (i.e. if \( T \in S, T \) has exactly 5 distinct elements of \( A \)). Prove:
There are distinct sets \( B \) and \( C \) in \( S \) such that
\[
\sum_{x \in B} x = \sum_{y \in C} y
\]
Help: \( \binom{10}{5} = 252 \)

The trick is to use the Pigeonhole Principle.
The number of 5-subsets of \( A \) is \( \binom{10}{5} = 252 \)
The number of possible sums of 5 positive integers, none greater than 50 is no more than \( 5 \times 50 = 250 \).
By the pigeonhole principle, there must be two 5-subsets with the same sum. \( \Box \)

It is possible to get a more accurate count of the values 5 elements can add up to.