1 The Pigeonhole Principle - Continued

1.1 Some Clever Applications

Example 10 in Rosen: in a 30-day month a baseball team plays at least one game every day, and no more than 45 games in the period. Then there exists a set of consecutive days $d_i, d_{i+1} \cdots d_{i+m}$, during which they played exactly 14 games.

Proof. Let $a_i$ be the number of games the team played up to and including day $i$: this yields a strictly increasing sequence $a_1, \cdots a_{30} \subset \{1, 2, \cdots 45\}$. In particular, this implies that the $a_j$ are distinct. Now consider the sequence $a_1 + 14, \cdots a_{30} + 14$. This sequence has also distinct elements. The union of the two sets of numbers is a subset of $\{1, 2, \cdots 59\}$, so there is a number that appears twice. Since each sequence is of distinct numbers, it must be that the pair belongs to different sequences. But then $\exists i, j : a_i = a_j + 14$, which means that from day $j + 1$ to day $i$ the team played exactly 14 games.
Example 1. This is example 11 in Rosen. Among any $n + 1$ numbers not exceeding $2^n$ there must be an integer that is a divisor of one of the other integers.

Proof. Let the integers be $a_1 \cdots a_{n+1}$. Write each as the product of an odd integer with a powers of 2, i.e. $a_i = q_i 2^n$, where $q_i$ is odd. Look at the set of the $q_i$: there are $n + 1$ elements in the set, but there are only $n$ odd numbers less than $2^n$—so there are two indices $i, j$, with $1 < j$ such that $q_i = q_j$. But then $a_i$ is a divisor of $a_j$. \qed

Finally, look at Theorem 3 in this Section of Rosen’s.

1.2 A taste of Ramsey Theory

Theorem 2. In a party with 6 people there are either

- a group of 3 people who have been introduced to each other, or
- a group of 3 people no two of whom were introduced to each other

Note: Rosen talk of ‘friends’ and ‘enemies’ which needs clarification—in real life these relations are not necessarily symmetric.

Proof. (see Rosen) is a relatively simple argument. Take a man, say A. Among the remaining 5, there must be a group of 3, say B, C, and D, who have either been all introduced to A, or none of whom were introduced (by Generalized Pigeonhole: 2 groups 5 people, one group must have 3 people.) WLOG, assume B, C, and D were introduced to A. If any two of them were introduced to each other, we are done. If not, we found a group of 3 no pair of whom were introduced to each other—and we are also done. \qed

This is the first step in a subject called Ramsey Theory. In general we have the following

Theorem 3. $\forall n, m > 0 \exists R(n, m)$ such that in any group of $R(n, m)$ people, there is either

- a group of $n$ people who have been introduced to each other, or
- a group of $m$ people no two of whom were introduced to each other

We will not cover the proof.
2 Combinations, Permutations, Binomial Coefficients

2.1 Permutations

Consider the following problem. You have a list of 10 excellent restaurants, and you get a discount coupon if you dine there on the last day of October, November and December. You can choose each restaurant only once. You wonder how many different dining experiences you can have, if your previous dinners influence your experience (for example if you went to the VeryExpensiveGourmetFrench and then you got to the FusionEthiopianJapanese, you will have a different experience than doing it in reverse order.) How many schedules can you create?

Solution: we have encountered this kind of counting before. You have 10 choices for the October date, then only 9 for the November one (you cannot repeat restaurants) and only 8 choice for December. We use the Product Rule to conclude that we have $10 \times 9 \times 8 = 720$ possible experiences among which we select one.

Definition 4. An $r$-permutation of $n$ objects is a sequence of length $r$ of distinct elements from the set of $n$ objects.

$P(n,r)$ is the number of $r$-permutations of $n$ elements.

We claim the $P(n,r) = n(n-1)(n-2) \cdots (n-r+1)$

This follows from formalizing the procedure we followed in solving the problem above: we get all sequences of length $n$ by choosing one of $n$ elements to be first, one of the remaining $n-1$ to be second, and finally one of the $n-[r-1] = n-r+1$ ones for the last. We apply the Product Rule to get the formula.

(Note: check that the last term is correct! We have $k$ ‘forbidden’ choices at stage $k$: we start with $k = 0$, and have $r$ stages, so the last number to subtract is $r - 1$. Another quick, very rough check is to plug in the numbers from the problem above—this will not be a proof, but will often reveal bad algebra.)

Perhaps you are familiar with the special case $P(n,n)$, often called simply permutation of $n$ elements where we have $P(n,n) = n!$ (n factorial). We shall use $0! = 1$.

Using this notation we have the formula
\[ P(n, r) = \frac{n!}{(n-r)!} \]

which you can verify directly (the \((n-r)!\) term of the denominator eliminates all but the first \(r\) terms of the product \(n!\).)

We give another combinatorial proof. A combinatorial proof of an identity usually consists in counting the same object two different ways.

**Claim 5.** \( P(n, r) = \frac{n!}{(n-r)!} \)

**Proof.** Let us count the sequences in \( P(n, r) \). Look at all \( n! \) permutations of \( n \) elements. We obtain them by first considering the first \( r \) elements: there are \( P(n, r) \) of these. We append to each of the sequences all permutations of the remaining \( n - r \) elements. There will be \( (n-r)! \) of there for each of the \( P(n,r) \) sequences. This construction generates all \( n! \) sequences. (Prove!! Why are we sure that every sequence is produced, and it is produced only once?)

So \( n! = P(n,r)(n-r)! \), which yields the required formula. A more realistic count takes into account that the marathon is almost always won by one of the elite runners invited to the event. There are only about 29 of those, so \( P(20, 3) \) is a more realistic estimate.

**Example 6.** Number of possible gold, silver and bronze medal winners in the Chicago Marathon. There are about 45,000 runners, so the number is \( P(45000, 3) \)

## 3 Combinations

**Definition 7.** \( C(n, r) \), is the number of subsets of cardinality \( r \) of a set with \( n \) elements.

**Claim 8.** \( C(n, r) = \frac{n!}{r!(n-r)!} \)

**Proof.** We will look again at \( n! \). On one hand, it is the set of permutations of \( n \) elements. Let us build them in a different way. Consider the first \( r \) positions: we get them by taking each of the \( C(n, r) \) \( r \)-subsets, producing the \( r! \) permutations of these, and concatenating to each all possible permutations of the remaining \( n-r \) elements. (Note that when we chose the first \( r \) we also
implicitly chose which are the $n-r$ remaining elements.) There are $(n-r)!$
possible such permutations for each of the first $r$ strings. So we have
$$n! = C(n, r)r!(n-r)!$$
which yields the desired formula.

**Example 9.** Number of (5-card) poker hands.

A deck has 4 suits of 13 cards each, a total of $4 \times 13 = 52$. So the answer
is $C(52, 5)$

**Example 10.** Number of full house hands.

A full house consists of a triple (like 3 aces, or 3 5s), and a pair. Note
that the pair and the triple must be different. (run away from a game where
a deck has 5 aces!)

We can count the number of full house cards by first counting the number
of ways to choose the kind of card that is the triples, and the kind that is the
pair. (For example, we could have AA A88 – we would have chosen A for
the triple and 8 for the pair.) There are $P(13, 2)$ ways to make this choice
(Why?) For the triple, we have $C(4, 3)$ choices, and for the pair $C(4, 2)$.

So the answer is $13 \times 12 \times \frac{4!}{3!2!} \times \frac{4!}{2!2!} = 1872$

if the undergrad does not change, how many new lines are possible?

There are very many similar counting problems, usually expressed in En-
lish (so that there is the problem of translating English into rigorous Math)
that you can solve with an ingenious mix of permutations and combinations.
Make sure you understand all examples in Rosen (try to solve them before
looking at the solution!) and solve many of the odd-numbered exercises.

**Claim 11.** $C(n, r) = C(n, n-r)$

*Proof.* (simple algebra) $C(n, r) = \frac{n!}{r!(n-r)!}$ and $C(n, n-r) = \frac{n!}{(n-r)!r!}$. Multi-
plication is commutative.

*Proof.* (combinatorial) Choosing an $r$-subset also chooses its complement.

A less cryptic version: there is a bijection between the subsets of card-
dinality $r$ and the sub sets of cardinality $n-r$: to each $r$-subset associate
its complement. (Prove this is a bijection!) SO the number of $r$-subsets,
$C(n, r)$ is the same as the number of $(n-r)$-subsets, $C(n, n-r)$. 

5
4 The Binomial Theorem

Definition 12. The symbol \( \binom{n}{k} \) denotes \( C(n, k) \). The symbols are called binomial coefficients.

Theorem 13. \( \forall n \geq 0 \) \( (a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \)

Proof. Expand \( (a + b)^n \) as
\[
(a + b)(a + b)\cdots(a + b)\cdots(a + b)
\]
\( n \) terms

Expanding the products, we see that the quantity is a sum of \( 2^n \) monomials, each of total degree \( n \), obtained by selecting one of the two terms (\( a \) or \( b \)) within each parenthesis. (Each monomial will be the product of \( n \) factors, \( i a \)s and \( j b \)s with \( i + j = n \).) Let us consider how many times the monomial \( a^{n-k} b^k \) appears.

We need to choose \( b \) in exactly \( k \) of the \( n \) terms of the product (this implies that we choose \( a \) in the remaining \( n-k \) terms.) The number of these choices is exactly \( \binom{n}{k} \) (we choose \( k \) indices from the positions \( \{1, 2, \cdots n\} \)).

The binomial theorem allows us to answer questions like: What is the coefficient of \( x^{12} y^{13} \) in the expression for \( (x + y)^{25} \)? (The answer is \( \binom{25}{12} \).)

You may ask ‘What if I wanted to chose the places for the \( y \) variable? It would yield \( \binom{25}{13} \)?’

The answer is that since \( \binom{n}{k} = \binom{n}{n-k} \) both answers are correct. In fact, the reasoning illustrates the proof of Claim 11.

The binomial theorem gives us some unexpected tricks:

Theorem 14. \( \sum_{k=0}^{n} \binom{n}{k} = 2^n \)

(i) Proof. (using Algebra)
\[
2^n = (1 + 1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^{n} \binom{n}{k}
\]

(ii) Proof. (combinatorial) The number of subsets of a set with \( n \) elements is \( 2^n \). The number of subsets is \( \sum_{k=0}^{n} \) (number of subsets of \( k \) elements).
But \( \binom{n}{k} \) is the number of subset \( s \) with \( k \) elements.

If you want to prove that an \( n \) element set has \( 2^n \) subsets, use previous knowledge of CS, and the Product Rule: We know that a subset of
an $n$-element $S$ set can be represented by a bit vector $v$ with of size $n$. $v[i] = 1$ iff the $i$-th element of $S$ is in the subset. This gives a 1-1 correspondence between bitstrings of length $n$ and subsets of $S$ (Prove formally!). We know (using the product rule and induction) that the number of bitstrings of length $n$ is $2^n$. 