1 Solving Recurrence Relations

In general, this is a hard problem, but we manage to do it most of the time.

2 Linear Recurrence Relations with Constant Coefficients

There is a special case, where there is a method which is gives us a solution in an almost mechanical way.

2.1 Homogeneous Linear Recurrence Relations with Constant Coefficients

Definition 1. A recurrence relation is linear if it is of the form

\[ a_n = \sum_{i=1}^k c_i a_{n-i} + \beta(n) \] where \( \beta(n) \) is a function \( \mathbb{N} \to \mathbb{R} \)

In other words, \( a_n \) depends linearly on the previous \( a_k \) \( k < n \).

We can rewrite the recurrence as \( a_n - \sum_{i=1}^k c_i a_{n-i} = \beta(n) \).

The recurrence relation is homogeneous if \( \beta(n) = 0 \).

Constant coefficients refers to the \( c_i \) - they are numbers.

For example the recurrence for the Fibonacci numbers

\[ f_i = f_{i-1} + f_{i-2} \] is a linear homogeneous relation with constant coefficients.
Some recurrences that are not:

\[ a_n = a_{n-1} - a_{n-2}^2 \] (not linear)
\[ a_n = 2a_{n-1} + 1 \] (not homogeneous). Note: this is the recurrence used in the Towers of Hanoi Problem.
\[ a_n = na_{n-1} \] (the coefficient of \(a_{n-1}\) is not a constant) Note: this is the recurrence used to define the factorial function.

2.1.1 Guessing a Solution

Consider the linear homogeneous recurrence with constant coefficients

\[ a_n - \sum_{i=1}^{k} c_i a_{n-i} = 0 \]

Let us take the (apparently) wild guess that there is a solution of the form

\[ a_n = x^n \]

where \(x\) is some (not yet determined) real number.

Let us try out our guess by substituting it in the recurrence relation. We have

\[ x^n - \sum_{i=1}^{k} c_i x^{n-i} = 0 \]

which we can rewrite as

\[ x^{n-k} [x^k - \sum_{i=1}^{k} c_i x^{k-i}] = 0 \]

Consider the expression

\[ x^k - \sum_{i=1}^{k} c_i x^{k-i} \]

Note that any value for \(x\) that satisfies the equality above will in fact be such that our guess \(a_n = x^n\) will satisfy the recurrence. Such values are exactly \(x = 0\), or \(x = r\), where \(r\) is a root of the polynomial \(x^k - \sum_{i=1}^{k} c_i x^{k-i}\).

This polynomial of degree \(k\), \(x^k - \sum_{i=1}^{k} c_i x^{k-i}\), is called the \textit{characteristic polynomial} of the recurrence.
We already knew that \( a_n = 0 \) for all \( n \) is a solution, but a very uninteresting one.

On the other hand, we can make an important observation.

**Lemma 2.** Let \( d_i \in \mathbb{R} \) and let \( r_1, r_2, \ldots, r_m \) be roots of the characteristic polynomial. Then \( a_n = \sum_{i=1}^{m} d_i r_i^n \) is a solution of the recurrence.

(Prove it! – Hint: \( 0 \cdot d_i = 0, \ 0 + 0 = 0 \))

Note that a sequence that satisfies the recurrence relation, does not solve our problem: we need to take into account also the *initial conditions* – the values of \( a_0, a_1, a_2 \) etc.

More precisely, assume that the characteristic polynomial has degree \( k \). By the Fundamental Theorem of Algebra, it has \( k \) roots. Assume all roots are distinct. Assume we have \( k \) values \( v_0, v_1, \ldots, v_{k-1} \) and we are given that

\[
\begin{align*}
  a_0 &= v_0 \\
  a_1 &= v_1 \\
  \quad \vdots \\
  a_{k-1} &= v_{k-1}
\end{align*}
\]

Then, for \( i = 1, 2, \ldots, k \) we can determine the coefficients \( d_1, d_2, \ldots, d_k \) that make the solution \( a_n = \sum_{i=1}^{m} d_i r_i^n \) satisfy both the initial conditions and the recurrence.

Note: if we succeed in doing this we will find *the* solution, since it must be unique: we know that given the values for \( a_0, \ldots, a_{k-1} \) we determine uniquely the values of \( a_n \), and the values so computed are exactly the ones we get from our solution.

To find these values of the \( d_j \)'s we plug in the values \( i = 0, 1, \ldots, i = k - 1 \) for the formula for \( a_i \)

\[
\begin{align*}
  a_0 &= d_1 r_1^0 + d_2 r_2^0 + \cdots + d_k r_k^0 \\
  a_1 &= d_1 r_1 + d_2 r_2 + \cdots + d_k r_k \\
  a_2 &= d_1 r_1^2 + d_2 r_2^2 + \cdots + d_k r_k^2 \\
  \quad \vdots \\
  a_{k-1} &= d_1 r_1^{k-1} + d_2 r_2^{k-1} + \cdots + d_k r_k^{k-1}
\end{align*}
\]
Turns out that this system of equations has a unique solution (the determinant of the systems is a Vandermonde determinant, whose value is $\prod_{i \neq j} (r_i - r_j)$ so if the roots are all distinct this determinant is nonzero.)

So we have the following

**Recipe for Solving Homogeneous Recurrence Relations**

1. Get the characteristic polynomial from the recurrence relation
2. Find the roots of the characteristic polynomial
3. *If all roots are distinct* all solutions will be of the form $a_n = \sum_{i=1}^{n} d_i r_i^n$
   where the $r_i$ are the distinct roots, and the $d_i$ are arbitrary constants
4. Determine the values of the $d_i$ by solving the system of linear equations that for $i = 0, 1, \cdots (k - 1)$ equates $a_i$ to the quantity given by the formula above for $i$.
5. Done!

**Example 3** (The Fibonacci recurrence). $f_n = f_{n-1} - f_{n-2}$

$f_0 = 0, f_1 = 1$

We rewrite the recurrence as

$f_n - f_{n-1} - f_{n-2} = 0,$

So the characteristic polynomial is

$x^2 - x - 1$

Its roots are

$r_1 = \frac{1+\sqrt{5}}{2}$ and $r_2 = \frac{1-\sqrt{5}}{2}$

So the general solution is

$f_n = d_1 \left( \frac{1+\sqrt{5}}{2} \right)^n + d_2 \left( \frac{1-\sqrt{5}}{2} \right)^n$
We find the values of $d_1$ and $d_2$ by solving the system of 2 equations

\[
\begin{align*}
0 &= d_1 \left( \frac{1+\sqrt{5}}{2} \right)^0 + d_2 \left( \frac{1-\sqrt{5}}{2} \right)^0 \\
1 &= d_1 \left( \frac{1+\sqrt{5}}{2} \right)^1 + d_2 \left( \frac{1-\sqrt{5}}{2} \right)^1
\end{align*}
\]

(0 and 1 are the values of $f_0$ and $f_1$ respectively)

The system of equations is simply

\[
\begin{align*}
0 &= d_1 + d_2 \\
1 &= d_1 \frac{1+\sqrt{5}}{2} + d_2 \frac{1-\sqrt{5}}{2}
\end{align*}
\]

They determine the values

\[
d_1 = \frac{1}{\sqrt{5}} \text{ and } d_2 = -\frac{1}{\sqrt{5}}
\]

which gives us the solution

\[
f_n = \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)^n
\]

### 2.1.2 Multiple Roots

Our recipe is useless if there are multiple roots. In the case of distinct roots, the recipe was (roughly) proven correct by our arguments. For the multiple root case, we will simply state how things work.

**Theorem 4.** If the characteristic polynomial has a root $r$ with multiplicity $m$, then $a_n = \sum_{i=0}^{m-1} e_i r^n$ is a solution of the recurrence for all values of the $e_i$.

In other words, instead of the single solution $r^n$, we have also $nr^n$, $n^2r^n$, all the way to $n^{m-1}r^n$ as solutions.

Again, we will have a general solution with undetermined parameters—we find values for them by using the general formula for $i = 0, 1, \cdots n-1$ and forcing the values to be $a_0, a_1, \cdots a_{n-1}$.
2.2 Nonhomogeneous Recurrence Relations

We will solve recurrence relations in the more general form

\[ a_n = \sum_{i=1}^{k} c_i a_{n-i} + \beta(n) \]

where \( \beta(n) \) is a function \( \mathbb{N} \to \mathbb{R} \).

Let \( a_n^{(p)} \) be a solution of the nonhomogeneous recurrence relation. (\( a_n^{(p)} \) has to satisfy the recurrence relation, but not the initial conditions!) We’ll call \( a_n^{(p)} \) a particular solution – it simply satisfies the recurrence (with the \( \beta(n) \) term), but we do not have the parameters to adapt it to the initial conditions.

**Lemma 5.** Let \( a_n^{(p)} \) be a particular solution of the nonhomogeneous recurrence relation, and let \( H(d_1, d_2, \cdots, d_m) \) be the (parametrized) general solution to the associated homogeneous recurrence (the recurrence where we set \( \beta(n) = 0 \)). Then

\[ a_n = H(d_1, d_2, \cdots, d_k) + a_n^{(p)} \]

is the general solution of the nonhomogeneous recurrence relation.

*(Sketch).* The homogeneous solution satisfies the characteristic polynomial (sets it to 0) and \( a_n^{(p)} \) takes care of the initial conditions.

So the recipe is essentially the same – except for the miracle needed to find a particular solution... Once we have it, we find the general solution of the homogeneous recurrence relation, using our recipe. We then add \( a_n^{(p)} \) to it, and determine the values of the parameters \( d_j \) using the initial conditions, as before. Rosen gives a recipe to find particular solutions in some special cases. We will simply present a complete example, where we sketch a method to find the solution.

**Example 6.** Find the general solution of the recurrence

\[ a_n = 5a_{n-1} - 6a_{n-2} + 7^n \]

. We first consider the homogeneous recurrence relation

\[ a_n = 5a_{n-1} - 6a_{n-2} \]

Its general solution is \( d_1 3^n + d_2 2^n \) (Solve it using the characteristic polynomial!)
Now we have to find a particular solution of the nonhomogeneous recurrence. An educated guess: try something like the nonhomogeneous part, the function $\beta(n)$.

So let us try to find a $C$ such that $C7^n$ satisfies the nonhomogeneous recurrence.

We substitute: $a_n = C7^n = 5C7^{n-1} - 6C^{n-2} + 7^n$

There is a leap of faith in the second equality – we do not know whether it can be made to hold, but we forge ahead...

Dividing both sides of the desired equality by $7^{n-2}$ we get

$$C7^2 = 5C7 - 6C + 7^2$$

which yields $C = 49/20$.

So we can be formal, righteous, and inscrutable by rewriting the solution as:

It is easy to see that if we consider the function $a_n = 7^n(49/20)$, it satisfies the nonhomogeneous recurrence relation, as it can be proven by substitution (and induction.)

Most times the strategy of trying a function that 'looks like' $\beta(n)$ will yield a good candidate for the nonhomegeneous particular solution. Once it is found, we can use our recipe to find a general solution, then find the values of the parameters that make the general solution satisfy the initial conditions.

2.3 Characterstic Equation with Multiple Roots

Read in Rosen. The general formula becomes more complicated: given a root $r$ of multiplicity $k$, instead of only getting the solution

$$a_n = r^n$$
we also get the solutions $a_n = r^n n, r^n n^2 \ldots a_n = r^n n^{k-1}$.

The proof is pushing symbols around, and neither we, nor Rosen do it (except for an exercise in Rosen for multiplicity 2)
2.4 A recipe for nonhomogeneous solutions

See Rosen. It is of limited utility.