1 Connectivity

One of the important properties of a graph is the ability (or lack thereof) of getting from one vertex to another, by traversing edges. We will need to define some formalism to make such notions precise. Unfortunately, there are differences between authors. We will use Babai’s terminology (cleaner, more precise, etc.) and point out how Rosen’s is different.

1.1 Definitions

A walk of length $k$ (in Rosen: path) is a sequence of vertices $v_0, v_1 \ldots v_k$, such that for $i = 0, 1, \ldots v_{k-1}$, $\{v_i, v_{i+1}\} \in E$.  

A trail (in Rosen simple path) is a walk with no repeated edges. 

A path (no corresponding definition in Rosen) is a walk with no repeated vertices. (Observe: a path is a trail, but not conversely.)

A path of length $k$ is denoted $P_{k+1}$ (Observe: the index is $k + 1$, not $k$! Why? what is the subscript counting?)

A closed walk (in Rosen circuit or cycle) of length $k$ is a sequence of vertices $v_0, v_1 \ldots v_k$ such that $v_0, v_1 \ldots v_{k-1}$ is a walk, $v_k = v_0$ and $\{v_{k-1}, v_0\} \in E$ (a walk, except $v_k = v_0$)
A cycle of length \( k \), or \( k \)-cycle (no corresponding definition in Rosen) is a closed walk of length \( k \) with no repeated vertices, except \( v_k = v_0 \). It is denoted by \( C_k \).

We have a corresponding set of definitions for digraphs: directed trail, directed path, directed cycle, etc. In these definitions the edges must be directed from \( v_i \) to \( v_{i+1} \).

**Definition 1. Vertices** \( x \) and \( y \) are connected if there is a path from \( x \) to \( y \). If \( \forall x, y \in V \) \( x \) and \( y \) are connected, we say that \( G \) is a connected graph.

**Definition 2.** A connected component of \( G \) is a maximal connected induced subgraph. That is, if \( H \) is a connected component, \( H \) is an induced subgraph (has all edges between it vertices that are edges of \( G \)), \( H \) is a connected graph, and it is maximal in the sense that if \( H' \) is any other connected subgraph of \( G \) that shares a vertex with \( H \), then \( H' \) is a subgraph of \( H \).

For the fans of Math:

An alternative definition is to note that \( x \) connected to \( y \) is an equivalence relation, and the connected components are the equivalence classes of this relation.

Exercise: Let \( x \sim y \) denote ‘\( x \) is connected to \( y \)’. Show that it is an equivalence relation - it has the properties:

- (reflexive): \( x \sim x \)
- (symmetric): if \( x \sim y \) then \( y \sim x \)
- (transitive): if \( x \sim y \) and \( y \sim z \) then \( x \sim z \)

So the connected component of a vertex \( x \) is the induced subgraph formed by the vertices \( y \) for which there is a path from \( x \) to \( y \).

For the CS fans:

From the sentence above it follows that one way of finding the connected component of \( x \) is to perform a Breadth First Search, starting at \( x \).

We note that connectivity could have been defined using walks instead of paths, because of the lemma below.
Lemma 3. If there is a walk from $x$ to $y$ there is a path from $x$ to $y$.

Proof. (Informal) Consider the walk from $x$ to $y$ (that exists by hypothesis.) If no vertex repeats, the walk is a path and we are done. Otherwise, consider the first repetition–so the walk is

$$x, v_1, \cdots v_i, v_{i+1}, \cdots v_{i+k} = v_i, v_{i+k+1} \cdots y$$

with $v_{i+k} = v_i$. Take out the repeating closed walk – i.e rewrite the walk as

$$x, v_1, \cdots v_i, v_{i+k+1} \cdots y.$$ 

Repeat the process, until we get a path – no repeating vertices. \qed

To make this argument into a proof we can use induction on the length of the walk. The induction hypothesis is that for all walks of length less than $k$ if there is a walk then there is a path that is no longer than the walk. This is certainly true for $k = 1$, and, for the induction step, if we have a walk of length $k$ the procedure above either yields a path or a shorter walk – which the induction hypothesis guarantees can be made into a path.

We will not study in detail connectivity in digraphs: the added difficulty is that ‘$y$ reachable from $x$’ (there is a directed path from $x$ to $y$) is not symmetric – the directed path from $y$ to $x$ may not exist. We consider instead the relation ‘$x$ is reachable from $y$ and $y$ is reachable from $x$’. The equivalence classes of this relation are the strong components of the digraph.

We will not look at further measures of connectivity covered in the Connectivity Section (10.4) of Rosen. One can study the number of vertices or the number of edges needed to increase the number of connected components of a graph—vertex connectivity, and edge connectivity, but we won’t.

2 Eulerian Cycles

There is an interesting history here – Euler’s paper describing his theorem is apparently the first paper in Graph Theory. It is on the course website (in the original Latin) as well as a translation. Rosen gives some of the interesting historical background. For history buffs, the history of the city of Königsberg/Kalinin with its changes through time may be interesting.

We will stick to Math. We could consider multigraphs (multiple parallel edges between 2 vertices allowed), which is needed for the Königsberg Bridges puzzle, but we won’t.
Definition 4. A graph is Eulerian if there is a closed walk with no repeated edges that includes all edges.

In other words, there is a walk that goes through every edge exactly once, and returns to the initial vertex.

Theorem 5. (Euler, 1736) A graph is Eulerian iff it is connected and every vertex has even degree.

Proof. If a graph is Eulerian, it must be connected. A walk that goes through every vertex arrives at it, then leaves again. This uses 2 distinct edges incident on the vertex, and these edges cannot be reused in the walk. If all edges are used, an even number was used at each vertex.

(The special case of a graph with a single vertex—of degree 0, which is even—also works...)

The converse is more involved, and will only be sketched. We choose a vertex \( v \) and start a walk. We claim that the walk must eventually return to \( v \) (for every other vertex, every time we enter, we make the number of remaining edges odd, so we may continue. The only vertex for which this may not be true is \( v \)—so we must return to it.) At this point we either have consumed all edges or not. If we did, we have our Euler Circuit (closed walk). If not, delete the edges of the walk. In the remaining graph there is a vertex, say \( w \), on the walk with nonzero (even) degree. Restart the process at \( w \), getting another closed walk, with edges unused in the first. We can put together these two walks into a larger walk, and continue the process. \( \square \)

As we (and Euler) argued, the same argument applies to multigraphs.

3 Hamiltonian Graphs

We will only deal with the concept, for completeness.

Definition 6. A graph is Hamiltonian if it has a spanning subgraph that is a cycle (Babai definition of cycle!)

In other words there is a closed walk which goes through every vertex exactly once.

Note that Hamiltonian differs from Eulerian only by changing edge to vertex. Since there is a good characterization of Eulerian graphs, and an
efficient algorithm to test whether a graph is Eulerian, one could conjecture that the same is true for deciding whether a graph is Hamiltonian.

One would be VERY wrong!

The decision problem is \textit{NP-complete}—not only there is no efficient algorithm for it, in spite of over a century of research, but it is widely conjectured (but not proven) that \textit{a polynomial time algorithm does not exist}. 