1 More Binomial Coefficients: Identities

These notes are going to be much shorter: Rosen covers almost all this material well, and often we will only give the outline of proofs and arguments.

1.1 The Binomial Theorem Trick

We have seen during the last lecture that substituting \( a = b = 1 \) in the formula for the Binomial Theorem

\[
\forall n \geq 0 \ (a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k
\]

we get that

\[
\sum_{k=0}^{n} \binom{n}{k} = 2^n
\]

Using the same trick, by setting \( a = 1, \ b = -1 \) we can get that

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0
\]

since the sum equals \((1 - 1)^n\).

An immediate corollary is that
\[ \sum_{k \text{ even}} \binom{n}{k} = \sum_{k \text{ odd}} \binom{n}{k} \]

(Corollaries 2 and 3 in Section 6.4)

The same trick yields
\[ \sum_{k=0}^{n} 2^k \binom{n}{k} = 3^k \]

(set \(a = 2, \ b = 1\))

### 1.2 Pascal’s Identity

**Theorem 1.** \( \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \)

We give two proofs: an algebraic one, and a combinatorial one.

(i) **Proof.** (using Algebra)
\[
\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!}
\]

Making both fractions have the same denominator, \(k!(n-k+1)\), we multiply both the numerator and the denominator of the first term by \(n - k + 1\), and the second term by \(k\). We get
\[
= \frac{n!k}{k!(n-k+1)!} + \frac{n!(n-k+1)!}{k!(n-k+1)!} = \frac{(n+1)!}{k!(n+1-k)!} = \binom{n+1}{k}
\]

(ii) **Proof.** (combinatorial)(sketch) \( \binom{n+1}{k} \) counts the number of \(k\)-subsets of \(n+1\) elements. The number of \(k\)-subsets element \(n+1\) is *not* in is \( \binom{n}{k} \) and the ones in which it *is* in is \( \binom{n-1}{k-1} \). See Rosen Theorem 2.

The identity can be used to build *Pascal’s Triangle*. See Rosen.

It is also a natural tool for proofs using induction...

### 1.3 Vandermonde’s Identity

**Theorem 2.** Let \( r \leq m, \ r \leq n; \ r, \ n, \ m > 0 \). Then \( \binom{n+m}{r} = \sum_{i=0}^{r} \binom{m}{r-i} \binom{n}{i} \)
The combinatorial proof considers two disjoint sets $A$ of cardinality $n$, and $B$, of cardinality $m$. We then count the $r$-subsets of $A \cup B$ two ways: one is the number of $r$-combinations of $n + m$ elements, the other adds the number of $r$-subsets with $i$ elements from $B$ and $r - i$ from $A$.

See Rosen Theorem 3 for details.

**Corollary 3.** \( \binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2 \)

**Proof.** Use Vandermonde with $n = m = r$ and use that \( \binom{n}{i} \binom{n}{n-i} = \binom{n}{i}^2 \) \[ \square \]

### 1.4 Last Identity

(last that is covered in class.)

**Theorem 4.** \( \binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r} \)

**Proof.** The LHS counts the number of bit strings of length $n + 1$ with exactly $r + 1$ ones (the $r + 1$ choices are the positions of the 1-bits.)

The RHS counts the same quantity as follows:

First choose the position of the rightmost $(r + 1)^{st}$ 1-bit. It must be some position $p \geq r + 1$ (or there are not enough positions for the remaining $r$ 1-bits). Let $j = p - 1$ be the position immediately to the left of the $(r + 1)^{st}$ 1-bit. The remaining $r$ bits can be put in positions $1, 2, \cdots j$ in $\binom{j}{r}$ ways. We sum over the possible values of $j$, as two different values obviously yield different bitstrings. \[ \square \]