Second Problem Set

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1. Lower Bounds for Sorting (Transposition)

   (a) Consider the following combinatorial problem: You are given $2b + 1$ bins, each capable of containing b balls. Initially, bins $B_1, B_2, \cdots B_b$ have b balls each, bins $B_{-b} \cdots B_0$ are empty. Balls come in one of b colors. Initially each nonempty bin has one ball of each color. The objective is to rearrange the balls so that each of bins $B_1, B_2, \cdots B_b$ be monochromatic (all the balls in a bin have the same color.

   The task is to be accomplished in *rounds*. There is a constant $k > 0$. In each round you are free to choose k bins, and freely exchange balls among them, subject to the capacity restriction.

   Prove that $\Theta(b \log b)$ rounds are necessary and sufficient to accomplish this.

   (Comment: This is an idealization of matrix transposition of a bxb matrix)

   (b) Now consider a $b(n)xb(n)$ matrix of $l(n)$ bit elements, with $n^{1/2} \log n \leq l(n) \leq n^{\alpha}$, for some $\alpha < 1$, and $b(n) = \sqrt{n/l(n)}$. The matrix will be stored in row order on the tape of a single tape k-head Turing machine. The l-bits long elements of the matrix will be denoted $w_{i,j}$, and a generic element will be denoted by w. The objective is to transpose the matrix.

      i. Given a Turing machine M that takes T steps to solve the problem, divide its tape into *blocks* of length $b(n)l(n)$, and its time into *periods* of length $b(n)l(n)$. A computation is *blockrespecting* if in each period each had stays entirely within a single block of tape. Show that there is a k', and a k'-head 1-tape Turing machine that is equivalent to M, and uses $O(T)$ steps.

      From now on we'll use b for $b(n)$, and $l$for $l(n)$. The idea will be to make each matrix entry an incompressible string, and use the argument for the balls.

      We use $C(x|y)$ to denote the Kolmogorov complexity of x given y, and $I(x|y) = length(x) - C(x|y)$.

      Let $B_i(t)$ be the contents of the i-th block at the end of the period t, and $V(t)$ the set of indices of the blocks visited during period t.

Let $A_ij(t) = \sum_{w \in B_j(t)} I(w|B_i(t))/l$
(Intuition: this would be the number of elements that should go to block B_j that are in B_i after period t)

ii. Prove: Given large enough n such that $b(n)$ is an integer, there are inputs $w = w_{1,1}, w_{1,2} \cdots w_{b,b}$ such that
$A_{i,j}(0) \leq 2$ for all i, j
$\sum_{w \in \ w_{1,1}, w_{1,2} \cdots w_{b,b}} I(w|B) \leq l(b+1)$for all $B \in \{0,1\}^{lb}$

Now assume that it is not the case that in any period t there i a $w \in w_{1,1}, w_{1,2} \cdots w_{b,b}$ such that
$\sum_{i \in V(t)} I(w|B_i(t) \geq \sum_{i \in V(t)} I(w|B_i(t-1) + l/b$
(”no magic occurs”)

iii. Let $K(t) = \sum_{i=-T}^{T} \sum_{j=1}^{b} A_{ij}(t) \log A_{ij}(t)$
Prove:
$K(0) \leq 2(2T+1)b$
$K(T) \geq \sum_j A_j j \log A_j j$ (remember the matrix is transposed after period T)
and $K(T) = \Omega(b^2 \log n)$

iv. Prove: $K(t) - K(t-1) \leq \sum_j S(j)$
where $S(j) = (\sum_{i \in V(t)} A_{ij}(t-1) + 1) \log(A_{ij}(t-1) + 1) - \sum_{i \in V(t)} A_{ij}(t-1) \log(1/2k \sum_{i \in V(t)} A_{ij}(t-1))$

v. Prove: $K(t) - K(t-1) = O(b)$

vi. Using (v) above, conclude that $T = \Omega(b \log b)$

2. Formalize and prove the following lower bound for on-line simulation of 2-dimensional tape by k 1-dimensional tapes.

   Write on the 2-dimensional tape a large incompressible square of N bits. Consider the position of the head on its 2-dimensional tape, and look at it as consisting of nonoverlapping nxn subsquares. If N and n are chosen well, the nxn subsquares will all be (almost) incompressible.

   Now consider a k (1-dimensional) tape simulator. Claim: there is a subsquare of the 2-dimensional tape that is ”far” from the 1-dimensional heads: while the 2-dimensional head can output it in time $O(\sqrt{(}N) + n^2$ (the first term is the time to get to it), the 1-dimensional machine will need $N + n^2$ moves. Repeating, we get a polynomial lower bound.

   State the theorem precisely, and prove the best bound you can.