MAX FLOW CHEAT SHEET

Digraph $G = (V,E)$

$s, t \in V$, $s \neq t$

some sink

Capacities $\forall e \in E$ $c_e \geq 0$

Flow $f: E \rightarrow \mathbb{R} \geq 0$

capacity constraints:

$\forall e \in E$ $0 \leq f(e) \leq c_e$

conservation constraints

$\forall u, v \neq s, t$ $\sum e=(v,u) f(e) = \sum e=(u,v) f(e)$

$f^{\text{in}} = f^{\text{out}} (s)$

$J(f) \equiv$ value of the flow $= \sum e=(v,u) f(e)$

$= f^{\text{out}} (s)$

RESIDUAL GRAPH: given $G, f$

$G_f$

for each $e \in E$

$\text{back edge}$

$\text{forward edge}$

$c_e - f(e)$

$\Rightarrow$

(capacity of back edge = flow from $u \rightarrow v$)

(residual capacity)

delete edges w/ capacity 0.

AUGMENTING FLOW

Given $G, f \Rightarrow G_f$

Find directed path from $s \rightarrow t$ in $G_f$; min capacity edge $b \leftarrow$ bottleneck $(P, f)$

Augmented flow $f_{\text{new}}(e) = \begin{cases} f(e) + b & \text{if } e=(u,v) \text{ & $P$ has the edge } (u,v) \text{ [forward edge]} \\ f(e) - b & \text{if } e=(v,u) \text{ & $P$ has the edge } (v,u) \text{ [back edge]} \\ f(e) & \text{otherwise} \end{cases}$
Max-Flow
Initially \( f(e) = 0 \) for all \( e \) in \( G \)
While there is an \( s-t \) path in the residual graph \( G_f \)
    Let \( P \) be a simple \( s-t \) path in \( G_f \)
    \( f' = \text{augment}(f, P) \)
    Update \( f \) to be \( f' \)
    Update the residual graph \( G_f \) to be \( G'_f \)
Endwhile
Return \( f \)

Properties:
1. At all times, before the execution of the loop, \( f(e) \), capacities in \( G_f \) edges are integers.
2. Augment increases \( v(f) \) by at least 1.
3. \( C = \sum_{e=(s,0)} c_e \) I the total capacity of edges leaving \( s \) is an upper bound on \( \text{max flow} \)

\[ \Rightarrow \text{Ford-Fulkerson terminates.} \]

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\( s-t \) cut: partition \( (A, B = V \setminus A) \) of \( V \) with \( s \in A \), \( t \in B \)

Capacity of \( s-t \) cut \( (A, B) \) \( \text{def.} = \sum_{e \text{ out of } A} c_e = c(A, B) \)

Claim: \( f \) \( s-t \) flow, \( (A, B) \) \( s-t \) cut

\[ v(f) = f^{\text{out}}(A) - f^{\text{in}}(A) \]

pf: \( v(f) = \sum_{v \in A} (f^{\text{out}}(v) - f^{\text{in}}(v)) = \sum_{v \text{ out of } A} f(e) - \sum_{v \text{ in } A} f(e) = f^{\text{out}}(A) - f^{\text{in}}(A) \)

Claim: \( f \) \( \text{ANY} s-t \) flow, \( (A, B) \) \( \text{ANY} s-t \) cut

\[ v(f) \leq c(A, B) \]

Corollary: if equal, both optimal
MAX FLOW MIN CUT THEOREM

When FF halts, no s-t paths in $G_f$

$$A^* = \{ v \in V : \exists \text{ directed s-t path in } G_f \} = \{ v \text{ reachable from } s \text{ in } G \}$$

$$B^* = V \setminus S$$

$(A^*, B^*)$ is an s-t cut.

Let $e = (u, v)$, $u \in A^*$, $v \in B^*$

$\uparrow$

edges of $G$

Claim: $f(e) = c_e$

Otherwise in $G_f$

$v$ reachable from $s$ in $G_f$

$u \in A^*$

Claim: $f(e) = 0$

Otherwise in $G_f$

$v$ reachable from $s$ in $G_f$

$f(e)$

$\leftarrow$

Back edge

So $\nu(f) = c(A^*, B^*)$

Thm: In every flow network the maximum value of an s-t flow is equal to the minimum capacity of an $s$-$t$ cut.

If capacities are integral, Ford-Fulkerson finds it.