# Extensions, Automorphisms, and Definability

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Abstract. This paper contains some results and open questions for automorphisms and definable properties of computably enumerable (c.e.) sets. It has long been apparent in automorphisms of c.e. sets, and is now becoming apparent in applications to topology and differential geometry, that it is important to know the *dynamical* properties of a c.e. set $W_e$, not merely *whether* an element $x$ is enumerated in $W_e$ but *when*, relative to its appearance in other c.e. sets.

We present here, and prove in another paper, a New Extension Theorem (N.E.T.) appropriate for $\Delta^0_3$-automorphisms of c.e. sets with simple hypotheses stated in terms of a dynamical concept called "templates" for the first time. The N.E.T. is needed implicitly or explicitly in almost every current automorphism proof for c.e. sets. We then present sketches of some known automorphism results using the N.E.T., and show how some definability results also derive their power from their ability to enforce a dynamic flow of elements into or away from a certain c.e. set.

To reveal the history and motivation for these results and questions, we give a detailed historical description in §1 and §2 which should be understandable to anyone who has read the first three or four chapters of Soare [1987]. The rest of the paper contains sketches of proofs which should be understandable to most computability theorists who have never worked in automorphisms. The open questions are chosen to illustrate a method or frontier we need to cross to advance our knowledge. Another paper on c.e. sets in this volume is Cholak [ta].

## 1. Introduction

**1.1. The Early History of C.E. Sets.** One of the great achievements of logic of the twentieth century was the development of the notion of a computable function. Foreshadowed by Gödel's use of the primitive recursive functions in his famous Incompleteness Theorem [1931], there soon emerged Gödel's definition [1934] of the general recursive functions. Shortly thereafter followed Turing's definition [1936] of functions computable by a Turing machine, and his argument that these constituted the intuitively computable functions. Gandy [1988, p. 82] observed,

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"Turing's analysis does much more than provide an argument for" Turing's Thesis, "*it proves a theorem.*" This definition and demonstration only then convinced Gödel of the Church-Turing Thesis that these comprised the class of all functions calculable by a human being with unlimited time and space. We accept this thesis and identify the terms "computable" and "Turing computable."

Almost simultaneously, Church and Kleene proposed a formal definition of an "effectively enumerable" set (the informal term for this concept), one which could be listed by an effective algorithm. This concept is probably second only to that of computable function for its importance in computability theory. These sets appear in many branches of mathematics (number theory, algebra, topology), and computer science. Actual digital computers produce such sets during their computations. Turing reducibility [1939] of one set to another can be defined by an effectively enumerable set of axioms.

Church [1936] and Kleene [1936] formally defined this notion as follows. A set $A \subseteq \omega$ is *computably enumerable (c.e.)* (previously called *recursively enumerable*) if $A$ is the range of a (Turing) computable function. (The empty set was added later as a c.e. set.) This is a *static* mathematical definition, like the definition of a function being continuous if the inverse image of every open set is open. Let $\{W_e\}_{e\in\omega}$ be the standard listing of the c.e. sets, as in Soare [1987].

Very little was heard about c.e. sets for half a decade. Then Post [1941] and [1943] introduced a *different* and unrelated formalism called a *(normal) production* system. Post's normal canonical system is a *generational* system, rather than a *computational* system as in general recursive functions or Turing computable functions. It led Post to concentrate on *effectively enumerable sets* rather than computable functions. He showed that every recursively enumerable set is a normal set (one derived in his normal canonical system) and therefore normal sets are formally equivalent to recursively enumerable sets.

Post used the terms "effectively enumerable set" and "generated set" almost interchangeably, for "the corresponding intuitive concept." Post [ 1944, p. 285] (like Church [1936]) formally defined a set of positive integers to be *computably enumerable* (formerly "recursively enumerable") if it is the range of a computable function. Post [1944], p. 286] explained his informal concept of a "generated set" ("effectively enumerable set") of positive integers this way,

> "Suffice it to say that each element of the set is at some time written down, and earmarked as belonging to the set, as a result of predetermined effective processes. It is understood that once an element is placed in the set, it stays there."

Post then [p. 286] restated his thesis from [1943] that *"every generated set of positive integers is recursively enumerable,"* [the italics are Post's] and he remarked that "this may be resolved into the two statements: every generated set is effectively enumerable, every effectively enumerable set of positive integers is recursively enumerable." Post continued, "their converses are immediately seen to be true." This is sometimes referred to as *"Post's Thesis."*

**1.2. Post Initiates the Study of C.E. Sets in 1944.** Explaining c.e. sets in terms of the quote above, and using his normal canonical system when necessary for mathematical precision, Post [1944] introduced several fundamental themes which have deeply influenced the subject ever since.

Theme 1.1. *Clear, informal style, using the Church-Turing Thesis.*

Up to that time, the papers by Kleene and others had been written in terms of the Gödel equational calculus or the Kleene schemata I–VI. (See Soare [1987, Ch. I].) Post explained everything in a very clear, informal style, using informal descriptions as we do today, and relying on Post's Thesis above or the Church-Turing Thesis to convert these, if necessary, to formal definitions of c.e. sets. This approach greatly stimulated interest in the topic of c.e. sets which had previously been largely ignored.

Theme 1.2. *(Dynamic) stage by stage computable construction of c.e. sets.*

Post moved significantly beyond the static conception of c.e. sets in Church and Kleene [1936] to a dynamic, stage by stage construction of c.e. sets, much like the process by which a modern digital computer performs its task. This is second nature to us now with our desktop computers, but it is not the way most mathematics was done in Post's time. It is far easier to follow than a formal mathematical definition in terms of Kleene's six schemata. From now on we use the term "dynamic" to refer to theorems which use a stage by stage construction in their proofs, but are not necessarily stated in dynamic terms as we discuss below, and we use the term "very dynamic" for those such as the New Extension Theorem 5.1 which use dynamic properties even in their statements.

Theme 1.3. *The study of reducibilities and relative computability.*

Post brought to a broad audience the definition of Turing reducibility from Turing [1939], where it had been obscurely placed in the middle of a long paper on another topic. Post augmented it by introducing several stronger reducibilities such as 1-reducible, m-reducible, tt-reducible, wtt-reducible, in addition to Turing's T-reducible. These were easier to understand than T-reducible, and Post related them to classes of c.e. sets he defined, such as creative, simple, h-simple, and hh-simple. His aim was to better understand how the information in a set $A$ can be coded into and recovered from another set $B$.

Theme 1.4. *Post's Problem.*

Post posed his famous *Post's Problem*, of whether there exists a c.e. set $A$, noncomputable but such that $A$ is strictly $T$-reducible to $K = \{e : e \in W_e\}$. In a subsequent paper, Post [1948], Post introduced the notion of two sets having the same *degree of unsolvability* if each is Turing reducible to the other. Thus, his problem can be stated as whether there is any degree of a c.e. set other than $\mathbf{0}$, the degree of the computable sets, and $\mathbf{0}'$, the degree of $K$. All this work on reducibilities, degrees, and coding of information had great impact.

Theme 1.5. *Relationship between set theoretic properties of c.e. sets and their degrees.*

Post approached his problem by trying to find a relationship between the (Turing) *degree* of c.e. set $A$ and its structure as a *set*, measured either with respect to set inclusion, or equivalently with respect to the epsilon relation for "being an element of." For example, Post defined a c.e. set $A$ to be *simple* if $\overline{A}$ is infinite, but contains no infinite c.e. set. Post gave a dynamic construction of a simple set and showed that such a set was incomplete with respect to a certain stronger reducibility, known as $m$-reducibility. We refer to *Post's Program* as the program to find connections between a c.e. set and its degree. This program had great influence, for example in the maximal set results by Yates, Sacks, and Martin in the 1960's and many later results.

It was only later that Myhill [1956] noticed that the collection of c.e. sets forms a lattice $\mathcal{E}$ under inclusion, but interestingly enough, almost all of the c.e. sets introduced by Post were later shown to be definable in the lattice $\mathcal{E}$. First, it is easy to see that the notion of being a finite set is definable (Soare, [1987, Ch X]), so it does not matter whether we consider $\mathcal{E}$ or $\mathcal{E}^*$, the lattice modulo finite sets. For $A \in \mathcal{E}$ let $A^*$ denote its equivalence class in $\mathcal{E}^*$.

Post's simple set is clearly $\mathcal{E}^*$-definable. Of Post's other sets, hh-simple and creative are $\mathcal{E}^*$-definable (as later proved by Lachlan and Harrington, respectively, see Soare [1987]), but only h-simple is not. *Post's Program* launched in his 1944 paper was to find connections between the set theoretic structure (usually the $\mathcal{E}$-structure) of a c.e. set $A$ and its (Turing) degree. This has had great influence on research ever since.

*Convention.* From now on all sets and degrees will c.e. unless otherwise stated.

## 2. Maximal Sets and Automorphisms

In 1956 Myhill called attention to the structure of the c.e. sets under inclusion, $\mathcal{E} = (\{W_e\}_{e\in\omega}, \subset)$, and noted that indeed it forms a lattice, although the lattice relations $\cup$, and $\cap$, and greatest element $\omega$, and $\emptyset$, are all definable over $\mathcal{E}$ from $\subset$, so to build an automophism, it suffices to deal only with inclusion. Myhill also asked whether there is a maximal set, namely a set maximal in the inclusion ordering modulo finite sets, $(\mathcal{E}^*, \subset^*)$.

### 2.1. Maximal Sets and the Structure of $\mathcal{E}$.

DEFINITION 2.1. A coinfinite c.e. set $A$ is *maximal* if there is no c.e. set $W$ such that $W \cap \overline{A}$ and $\overline{W} \cap \overline{A}$ are both infinite. Equivalently, $A$ is maximal iff $A^*$ is a coatom (maximal element) in $\mathcal{E}^*$.

At the large and memorable Cornell logic meeting in 1957, Friedberg presented his construction of a maximal set [1958], done shortly after his solution of Post's problem [1957]. Maximal sets then became the object of intense study for several reasons: they were coatoms of $\mathcal{E}^*$ and hence the building blocks for more complicated lattices of supersets, they were the ultimate realization of Post's search for sets with thin complements; their degrees were very interesting, as we now see.

Tennenbaum conjectured that every maximal set is complete, but Sacks [1964] refuted this by building an incomplete maximal set. Yates [1965] built a complete maximal set. Martin completed this thread with a beautiful characterization of the Turing degrees (information content) of maximal sets.

DEFINITION 2.2. A c.e. set $A$ is *high* (*low*) if its Turing jump $A' \equiv_{\mathrm{T}} \emptyset''$ ($A' \equiv_{\mathrm{T}} \emptyset'$), and the degree of a set $A$, written $\deg(A)$, is high or low according as $A$ is. (See Soare [1987, Ch. III].)

THEOREM 2.3 (Martin, 1968). *The degrees of maximal sets are exactly the high degrees.*

Extending from maximal sets to those with the next thinnest complements, the hh-simple sets, Lachlan then proved a corresponding theorem.

Definition 2.4. (i) Let $\mathcal{L}(A) = \{W : A \subseteq W\}$ the lattice of supersets of $A$.
(ii) Let $\mathcal{E}(S) = \{W \cap S : W \text{c.e. }\}$ the lattice of c.e. sets restricted to $S$, where $S$ is not necessarily c.e.

Notice that for $A$ c.e. there is a natural isomorphism between $\mathcal{L}(A)$ and $\mathcal{E}(\overline{A})$.

Theorem 2.5 (Lachlan, 1968). *(i) A c.e. set $A$ is hh-simple iff $\mathcal{L}(A)$ is a Boolean algebra.*
*(ii) The degrees of hh-simple sets are exactly the high degrees.*

For (ii) it had been known that hh-simple sets are high. Lachlan used Martin's method to prove that for every hh-simple set $B$ and every high degree $\mathbf{d}$, there is an hh-simple $A \in \mathbf{d}$ such that $\mathcal{L}(A) \cong \mathcal{L}(B)$, which is stronger than (ii).

**2.2. Felix Klein's Erlanger Programm.** Another mathematical predecessor of the present paper is the emphasis on automorphism of a structure introduced by Felix Klein. When he was appointed to his professorship at the University of Erlangen his inaugural speech was devoted to a new approach to geometry. He argued that one should characterize a structure by those properties which remain invariant under all transformations which preserve the structure. This had a profound effect on the development of geometry.

In computability theory, by the later 1960's the structure of $\mathcal{E}$ was better understood, and people began asking about the automorphisms of $\mathcal{E}$. Several of these questions were stated in Rogers book [1967] such as: "Is every automorphism of $\mathcal{E}^*$ induced by one of $\mathcal{E}$?" "Is creativity invariant under $Aut(\mathcal{E})$?" Martin and Lachlan asked whether any two maximal sets are automorphic, namely are they in the same orbit?

**2.3. Static Automorphisms in Mathematics.** Automorphisms of a given structure now play a very important role in many branches of mathematics, for example, group theory, field theory and many others. When an algebraist builds an automorphism $\Phi$ of a countable object such as a group $\mathcal{G} = \{g_0, g_1, g_3, \dots\}$, he usually starts with an element $g_0$, picks out a suitable image $g_{h(0)}$ and continues. Even if he does not use a step by step approach, he still views the elements of $\mathcal{G}$ as indivisible whole objects. He usually does not collect some information on "part of" $g_n$ during the process as we do here.

Since $\mathcal{E} = (\{W_n\}_{n\in\omega}, \subset)$ is so complicated, here we have no idea how to pick out for an arbitrary $W_n$ an image $W_{h(n)}$ of the same elementary 1-type, much less how to continue the process to make $\Phi$ total, onto, and an automorphism. Hence, we are forced to take the partial information offered about $W_{n,s}$ during the construction and use it to build our approximation $\widehat{W}_{n,s}$ at stage $s$ to the image $\widehat{W}_n = \Phi(W_n)$.

**2.4. Dynamic Automorphisms for C.E. Sets.** Every computable permutation of $\omega$ produces a trivial automorphism of $\mathcal{E}$, but the first nontrivial automorphism was constructed by Martin (unpublished, see Soare [1987, p. 345]) who showed that hypersimplicity is noninvariant, the only one of Post's properties to be noninvariant. A much more complicated method was necessary for more general automorphisms and was given by Soare who answered the preceding Martin-Lachlan question with the next result.

Theorem 2.6 (Soare, 1974). *If $A$ and $B$ are any two maximal sets then there is an automorphism $\Phi$ of $\mathcal{E}$ mapping $A$ to $B$.*

We defer a detailed discussion of the proof until §6, but we give here a little intuition. To approach this problem we cannot use a static automorphism, like the algebraist, but rather we need a *dynamic* approach to building the automorphism $\Phi$. Given a set $U$ we must build a set $\widehat{U}$ for $\Phi(U)$, enumerating elements in $\widehat{U}$ as more and more are enumerated in $A$, $B$, and $U$.

For each automorphism theorem we fix some simultaneous computable enumeration of all the sets under consideration, and let $X_s$ denote the finite set of elements enumerated in $X$ by the end of stage $s$. With respect to this simultaneous enumeration, define

$$X \setminus Y = \{x : (\exists s)[x \in X_s - Y_s]\}, \tag{1}$$

$$X \searrow Y = (X \setminus Y) \cap Y. \tag{2}$$

If elements $x$ enter $U \setminus A$, threatening to make $|U - A| = \infty$, then we must enumerate some elements $y$ into $\widehat{U} \setminus B$ toward making $|\widehat{U} \setminus B| = \infty$ also. Now the opponent can move such $y$ into $B$ ensuring $|\widehat{U} \searrow B| = \infty$. But we must match each such $y$ with an $x \in U \cap A$ (and the opponent may arrange that $|A \searrow U| = \emptyset$.) Hence, in general we will need to guarantee that we can satisfy a *covering property* such as:

$$|\widehat{U} \searrow B| = \infty \implies |U \searrow A| = \infty. \tag{3}$$

To see the necessity of a condition like (3), suppose that $B$ is simple and $A$ is nonsimple with an infinite set $U \subset \overline{A}$. Now $U \setminus A$ infinite forces $\widehat{U} \setminus B$ infinite, but the opponent enumerates $B$ to force $\widehat{U} \searrow B$ infinite, while $U \cap A = \emptyset$.

This is only a very weak version of the covering that is needed, and then after that we need an extension theorem like Theorem 5.1 to guarantee that the covering is sufficient to produce an automorphism, as we develop in §5.

The main thing to take away from this section is that dynamically building an automorphism from say, $U_1, \ldots U_n$, to $\widehat{U}_1, \ldots, \widehat{U}_n$, involves moving elements around on a giant Venn diagram with $2^n$ pieces, and trying to guarantee that one piece (state) is *well-resided* (occupied by infinitely many permanent residents) iff the corresponding piece on the other side is well-resided. The conditions like (3), (1), and (2), often involve the dynamic *flow* of elements from one well-visited state to another in the Venn diagram, where a piece (state) is *well-visited* if infinitely many elements rest there at least temporarily.

Definition 2.7. We say that a construction like Post's is *dynamic* if it involves a stage by stage construction, like most in computability theory, and *very dynamic* if it requires notions like (3), (1), and (2) which involve the flow of elements.

We will examine this in more detail later. However, with this brief glimpse at very dynamic properties, let us look at the surprising role they play in definability results.

## 3. Very Dynamic Aspects of Definable Properties.

For properties like Post's simple or hh-simple set, the definition and consequences are entirely static even though the construction may be dynamic. In the last two decades there arose a new class of proofs which relate a property $P(A)$, which is $\mathcal{E}$-definable (and therefore entirely static), to other static aspects such as $\deg(A)$, but the connection is entirely via certain *very dynamic* properties regarding the flow in a Venn diagram as above.

During the 1970's and early 1980's researchers had been trying to use the automorphism method to show that creative sets were not invariant, by taking a creative set $A$ to a noncreative set $B$ by an automorphism. Harrington analyzed the failure of these attempts and started an entirely new line of $\mathcal{E}$-definable properties, whose static definitions force some very dynamic behavior.

**3.1. The Defining Property for Creative Sets.** Post [1944] defined a c.e. set $C$ to be *creative* if there is a partial computable function $\psi$ such that

$$(\forall e)[W_e \subset \overline{C} \implies \psi(e)\downarrow \in \overline{C} - W_e]$$

It follows by Myhill's Theorem (see Soare [1987, p. 43]) that $C$ is creative iff $C$ is $m$-complete, *i.e.*, $W_e \leq_m C$ for every $e$, or equivalently $K \leq_m C$ for the Gödel complete set $K$. Although these properties of $C$ at first appear to be very far from being $\mathcal{E}$-definable, Harrington (see Soare [1987, p. 339]) exhibited the following $\mathcal{E}$-definable property CRE(A) which defines $C$ being creative.

THEOREM 3.1 (Harrington). *A c.e. set $A$ is creative iff*

$$\text{CRE(A)}: \quad (\exists C \supset A)(\forall B \subseteq C)(\exists R)[R \textit{ is computable} \tag{4}$$

$$\& \;\; R \cap C \textit{ is noncomputable} \;\;\&\;\; R \cap A = R \cap B], \tag{5}$$

*where all variables range over $\mathcal{E}$.*

We may represent the property CRE($A$) as a two person game in the sense of Lachlan [1970] between the $\exists$-player (called RED, the definability player) who plays the c.e. sets $C$, $R$ (the red sets) and the $\forall$-player (called BLUE, the automorphism player) who plays the c.e. set $B$ (the blue set).

THEOREM 3.2 (Blue). *If* CRE($A$) *then $K \leq_m A$ so $A$ is creative.*

PROOF. (Sketch). Suppose CRE($A$). We may visualize $R$ as dividing the universe $\omega$ into two halves and on the $R$ half we visualize in the Venn diagram the following states (corresponding roughly to $e$-states) $\nu_1 = R \cap \overline{C}$, $\nu_2 = R \cap (C - B)$, $\nu_3 = R \cap (B - A)$, $\nu_4 = R \cap A$. The static condition CRE($A$) forces certain dynamic properties of the sets as follows. The condition that $R \cap C$ is noncomputable means that $R - C$ is not c.e. so there must be an infinite c.e. set of elements, say $\{x_n\}_{n \in \omega}$, which move from state $\nu_1$ to state $\nu_2$. Define $\psi(n) = x_n$. If $n$ enters $K$, then enumerate $\psi(n)$ in $B$, from which the second conjunct of (5) eventually forces that $\psi(n) \in A$, so $x_n$ passes from $\nu_1$ to $\nu_2$ to $\nu_3$ to $\nu_4$ in that order. If $n \in K$ then $x_n$ remains in $\nu_2$ forever. Hence, $K \leq_m A$ via $\psi$. For more details see Harrington-Soare [1998]. $\square$

**3.2. The Property $Q(A)$ Guaranteeing Incompleteness.** In 1991 Harrington and Soare gave an $\mathcal{E}$-definable solution to Post's problem by producing an $\mathcal{E}$-definable property $Q(A)$ which guarantees that $A$ is noncomputable and incomplete. It produces a much slower dynamic flow into $A$ which prevents $A$ from being prompt and hence complete.

Definition 3.3. (i) A coinfinite c.e. set $A$ is *promptly simple* if there is a computable function $p$ and a computable enumeration $\{ A_s \}_{s\in\omega}$ of $A$ such that for every $e$,

$$W_e \text{ infinite} \Longrightarrow (\exists s)\ (\exists x)\ [x \in W_{e,\text{ at } s} \cap A_{p(s)}]. \tag{6}$$

(ii) An c.e. set $A$ is *prompt* if $A$ has promptly simple degree namely, $A \equiv_T B$ for some promptly simple set $B$, and an c.e. degree is *prompt* if it contains a prompt set.

(iii) An c.e. set or degree which is not prompt is *tardy.*

By the Promptly Simple Degree Theorem, see Theorem XIII.1.7(iii) of Soare [1987], a set $A$ being prompt is equivalent to the following property. Let $\{A_s\}_{s\in\omega}$ be any computable enumeration of $A$. Then there is a computable function $p$ such that for all $s$, $p(s) \geq s$, and for all $e$,

$$W_e \text{ infinite} \quad\Longrightarrow\quad (\exists^\infty x)\ (\exists s)\ [x \in W_{e,\text{ at } s}\ \ \&\ \ A_s{\restriction} x \neq A_{p(s)}{\restriction} x], \tag{7}$$

namely infinitely often $A$ "promptly permits" on some element $x \in W_e$.

Definition 3.4. (i) A subset $A \subset C$ is a *major subset* of $C$(written $A \subset_{\mathrm{m}} C$) if $C - A$ is infinite and for all $e$,

$$\overline{C} \subseteq W_e \implies \overline{A} \subseteq^* W_e.$$

(Note that if $A \subset_{\mathrm{m}} C$ then both $A$ and $C$ are noncomputable.)

(ii) $A \sqsubset B$ if there exists $C$ such that $A \sqcup C = B$ (i.e. $A \cup C = B$ and $A \cap C = \emptyset$).

Theorem 3.5 (Harrington-Soare, 1991). *There is a property $Q(A)$ which guarantees that A is tardy and hence, $A <_{\mathrm{T}} K$, and which holds of some noncomputable set.*

Define the property: $\quad Q(A): \quad (\exists C)_{A\subset_{\mathrm{m}}C}\ (\forall B \subseteq C)(\exists D \subseteq C)(\forall S)_{S\sqsubset C}\ [$

$$[B \cap (S - A) = D \cap (S - A)] \tag{8}$$

$$\Longrightarrow (\exists T)[\overline{C} \subset T\ \&\ A \cap (S \cap T) = B \cap (S \cap T)]]. \tag{9}$$

Theorem 3.6. *If $Q(A)$ then $A$ is incomplete (*i.e.*, $A \not\leq_{\mathrm{T}} K$).*

Proof. (Sketch only, see Harrington-Soare [1991] for details.) We may visualize the property $Q(A)$ as a two person game in the sense of Lachlan [1970] between the $\exists$-player (RED) who plays the c.e. sets $A$, $C$, $D$ and $T$ and the $\forall$-player (BLUE) who plays the c.e. sets $B$ and $S$. For simplicity ignore all the sets but $C$, $D$, $B$, and $A$, since the others are necessary only to give us a suitable domain on which to play the following strategy. Visualize $C \supseteq D \supseteq B \supseteq A$, and let $\nu_1, \nu_2, \ldots \nu_5$ denote the differences of c.e. sets (called *d.c.e. sets*): $\omega - C$, $C - D$, $D - B$, $B - A$, $A$ respectively, but viewed dynamically like $e$-states, so an element can pass from $\nu_i$ to $\nu_j$, $i < j$. The oversimplified $Q(A)$ property now asserts that if BLUE plays: $(8)'$ $D = B$ on $\overline{A}$, then RED will play: $(9)'$ $B = A$. In particular, if both players are following their best strategies, then for an element $x$ to enter $A$, it must pass through the $\nu$-states in the order $\nu_1, \nu_2, \ldots \nu_5$ as proved in Harrington-Soare [1991]. However, the set $B$ acts like a wall of restraint, like the minimal pair restraint of Lachlan and Yates in Soare [1987, p. 153]. When presented with an $x \in D - B$, BLUE may hold $x$ as long as he likes, but must eventually put $x$ into $B$ at which point RED is free to put $x$ into $A$ but not before. This implies that $A$ is tardy (*i.e.*, not of promptly simple degree, see Ambos-Spies, Jockusch, Shore, and Soare in Soare [1987, p. 284] or [1987, Chap III], so $A$ is incomplete. $\square$

Furthermore, Harrington-Soare [1996b] have discovered that $Q(A)$ imposes a much stronger tardiness property on $A$ (called *2-tardy*) which helps us classify those sets which can be coded into any nontrivial orbit.

There are other $\mathcal{E}$-definable properties such as $T(A)$ in Harrington-Soare [1998] which guarantee that $A$ is complete and which can hold of promptly simple sets. This works by ensuring the existence of a state $\nu_1$ (of elements) in $\overline{A}$ which is well resided, but from which it is legal to move any element to some $\nu_s$ inside of $A$ for coding $K \leq_{\mathrm{T}} A$. There is also a property $NL(A)$ which ensures that $A$ is not low even though it can be low$_2$ and promptly simple. It achieves this by forcing an infinite stream of elements to move through a sequence of states to overcome any function which is a candidate to prove lowness. (See Harrington-Soare [1988] on $NL(A)$ for details.)

The conclusion to draw is that not only do definable properties act in opposition to automorphisms, each limiting the power of the other, but now the battlefield on which they compete is that of very dynamic properties, an arena not obvious from the static definitions of $\mathcal{E}$-definable properties.

## 4. Building an Automorphism of $\mathcal{E}$

We fix two copies of the integers, $\omega$ and $\widehat{\omega}$ and a standard listing $\{U_n\}$ of the c.e. sets on the $\omega$-side and $\{V_n\}$ on the $\widehat{\omega}$-side, which we view as being played by the opponent, called RED . During our dynamic construction we, Player BLUE, must construct sets $\{\widehat{U_n}\}$ for the $\{\widehat{\omega}\}$-side, and $\{\widehat{V_n}\}$ for the $\omega$-side which meet the following condition, (10) and usually the stronger condition (11).

### 4.1. Definitions of e-states.

Definition 4.1. (i) Given two sequences of c.e. sets $\{X_n\}_{n\in\omega}$ and $\{Y_n\}_{n\in\omega}$, define $\nu(e,x)$, the *full e-state* of $x$ with respect to (w.r.t.) $\{X_n\}_{n\in\omega}$ and $\{Y_n\}_{n\in\omega}$ to be the triple $\langle e, \sigma(e,x), \tau(e,x)\rangle$, where

$$\sigma(e,x) = \{i : i \leq e \ \ \& \ \ x \in X_i\}, \quad \tau(e,x) = \{i : i \leq e \ \ \& \ \ x \in Y_i\}.$$

(ii) If $x \in \omega$ we measure $\nu(e,x)$ with respect to $\{U_n\}_{n\in\omega}$ and $\{\widehat{V}_n\}_{n\in\omega}$, and if $\hat{x} \in \widehat{\omega}$ we measure $\widehat{\nu}(e,\hat{x})$ with respect to $\{\widehat{U}_n\}_{n\in\omega}$ and $\{V_n\}_{n\in\omega}$. If $\nu(e,x) = \nu$ we say $x$ is *in* $e$-state $\nu$ and likewise for $\hat{x}$ and $\widehat{\nu}(e,\hat{x})$.

(iii) If $U_{n,s}$, $\widehat{V}_{n,s}$, $n, s \in \omega$ is a computable approximation to $U_n$, $\widehat{V}_n$, then we define $\nu(e,x,s)$ as above, but with $U_n$, $\widehat{V}_n$ replaced by $U_{n,s}$, $\widehat{V}_{n,s}$ and likewise for $\widehat{\nu}(e,\hat{x},s)$ with respect to $\widehat{U}_{n,s}$ and $V_{n,s}$.

Definition 4.2. (i) The *well-resided* $e$-states on the $\omega$-side and $\widehat{\omega}$-side respectively are

$$\mathcal{K}_e = \{\nu : (\exists^\infty x)[\nu(e,x) = \nu]\}, \quad \widehat{\mathcal{K}}_e = \{\nu : (\exists^\infty \hat{x})[\widehat{\nu}(e,\hat{x}) = \nu]\}.$$

(ii) The *well-visited* states are

$$\mathcal{M}_e = \{\nu : (\exists^\infty x)[\nu(e,x,s) = \nu]\}, \quad \widehat{\mathcal{M}}_e = \{\nu : (\exists^\infty \hat{x})[\nu(e,\hat{x},s) = \nu]\}.$$

(iii) For the $\omega$-side the *well-resided* states are $\mathcal{K} = \bigcup_{e\in\omega} \mathcal{K}_e$, the *well-visited states* are $\mathcal{M} = \bigcup_{e\in\omega} \mathcal{M}_e$, and similarly define $\widehat{\mathcal{K}}$ and $\widehat{\mathcal{M}}$ for the $\widehat{\omega}$ side.

The picture is now very simple. The $e$-states $\nu$ measure boxes in the Venn diagram of $\omega$ partitioned by $U_n$, $\widehat{V}_n$, $n \leq 3\ e$, giving $2^{2e+2}$ partitions or full $e$-states, and likewise for the $\widehat{\omega}$-side. To ensure that $\Phi$ is an automorphism it is necessary and sufficient to ensure that every full $e$-state $\nu$ is well-resided on the $\omega$-side iff the corresponding full $e$-state $\widehat{\nu}$ is well-resided on the $\widehat{\omega}$-side, namely iff

$$\mathcal{K} = \widehat{\mathcal{K}}. \tag{10}$$

In practice, we achieve (10) by achieving the somewhat stronger condition,

$$\mathcal{M} = \widehat{\mathcal{M}} \quad \text{and} \quad \mathcal{N} = \widehat{\mathcal{N}}, \tag{11}$$

where the set $\mathcal{N}$ ($\widehat{\mathcal{N}}$) consists of those $\nu \in \mathcal{M}(\widehat{\mathcal{M}})$ which are "emptied out" during the construction in the sense that almost all elements in $\nu$ are moved, either by RED or BLUE, to another state. Clearly, the well-resided states $\mathcal{K}$ are exactly those states $\nu$ which are well-visited ($\nu \in \mathcal{M}$) but not emptied out ($\nu \notin \mathcal{N}$). Hence,

$$\mathcal{K} = \mathcal{M} - \mathcal{N} \ \ \& \ \ \widehat{\mathcal{K}} = \widehat{\mathcal{M}} - \widehat{\mathcal{N}}. \tag{12}$$

Thus, the primary strategy for the automorphism player, BLUE, is to *copy* the moves of RED, namely first to copy the well visited states to make $\mathcal{M} = \widehat{\mathcal{M}}$ and then to empty out a state if RED empties out the corresponding state.

**4.2. Building an Automorphism on a Tree.** If we proceed as in §4.1 as for Soare's original proof of Theorem 6.5 we produce an effective automorphism in the sense that the automorphism is presented by a computable map on indices of c.e. sets. It is more powerful to combine the automorphism method with Lachlan's method of trees to produce, not effective automorphisms, but $\Delta^0_3$-automorphisms as introduced by Harrington-Soare [1996c] and Cholak [1995]. These are much more versatile, as we shall see.

The priority tree $T$ consists of nodes defined roughly as follows. Suppose node $\beta \in T$. We put nodes $\alpha = \beta\widehat{\ }\langle k_\alpha, \mathcal{M}_\alpha, \mathcal{N}_\alpha \rangle$ in $T$ where sets $\mathcal{M}_\alpha$ and $\mathcal{N}_\alpha$ have approximately the meanings above, and $k_\alpha$ is the least $n$ beyond which all elements move only among well-visited states. We let $f$ denote the true path of $T$ and we use a computable approximation $f_s$ such that $f = \liminf_s f_s$.

Each element $x$ of the $\omega$-side is placed at the end of stage $s$ on a node $\alpha \in T$ denoted by $\alpha(x, s)$. Define the c.e. set

$$Y_{\alpha,s} = \{x : (\exists t \leq s)[\alpha(x,t) \supseteq \alpha]\}, \tag{13}$$

where $\gamma \supseteq \beta$ denotes that $\gamma$ is an extension of $\beta$ on $T$. We omit most of the details of $T$ which can be found in Harrington-Soare [1996c, §2].

**4.3. $\alpha$-states.** For conceptual simplicity we do as little action as possible at each node $\alpha \in T$. If $|\alpha| \equiv 1 \bmod 5$ ($|\alpha| \equiv 2 \bmod 5$), we consider one new $U$ set ($V$ set). If $|\alpha| \equiv 3 \bmod 5$ ($|\alpha| \equiv 4 \bmod 5$), we consider new $\alpha$-states $\nu$ ($\hat{\nu}$) which may be non well-resided on $Y_\alpha$ ($\widehat{Y}_\alpha$). If $\alpha \equiv 0 \bmod 5$ we make no new commitments for the automorphism machinery but we may perform action for some additional requirement (such as coding information into $B$). We shall arrange that for all $n \in \omega$ that for $\alpha \subset f$,

$$|\alpha| = 5n + 1 \implies U_\alpha =^* U_n, \text{ and} \tag{14}$$

$$|\alpha| = 5n + 2 \implies V_\alpha =^* V_n. \tag{15}$$

We let $U_\alpha$ and $\widehat{U}_\alpha$ ($V_\alpha$ and $\widehat{V}_\alpha$) be undefined if $|\alpha| \not\equiv 1 \bmod 5$ ($|\alpha| \not\equiv 2 \bmod 5$). We let $e_\alpha(\hat{e}_\alpha)$ correspond to $n$ in (14) (respectively (15)). Namely, define $e_\lambda = \hat{e}_\lambda = -1$ and if $|\alpha| \equiv 1 \bmod 5$ then let $e_\alpha = e_{\alpha^-} + 1$, and otherwise let $e_\alpha = e_{\alpha^-}$. Define $\hat{e}_\alpha$ similarly with $|\alpha| \equiv 2 \bmod 5$ in place of $|\alpha| \equiv 1 \bmod 5$. Hence, $e_\alpha > e_{\alpha^-}$ ($\hat{e}_\alpha > \hat{e}_{\alpha^-}$) iff $|\alpha| \equiv 1 \bmod 5$ ($|\alpha| \equiv 2 \bmod 5$).

Definition 4.3. An *$\alpha$-state* is a triple $\langle \alpha, \sigma, \tau \rangle$ where $\sigma \subseteq \{0, \dots, e_\alpha\}$ and $\tau \subseteq \{0, \dots, \hat{e}_\alpha\}$. The only $\lambda$-state is $\nu_{-1} =< \lambda, \emptyset, \emptyset \rangle$.

The construction will produce a simultaneous computable enumeration $U_{\alpha,s}$, $V_{\alpha,s}$, $\widehat{U}_{\alpha,s}$, $\widehat{V}_{\alpha,s}$, for $\alpha \in T$ and $s \in \omega$, of these r.e. sets which we use in the following definition.

Definition 4.4. (i) The *$\alpha$-state of $x$ at stage $s$*, $\nu(\alpha, x, s)$, is the triple $\langle \alpha, \sigma(\alpha, x, s), \tau(\alpha, x, s) \rangle$ where

$$\sigma(\alpha, x, s) = \{e_\beta : \beta \subseteq \alpha \ \& \ e_\beta > e_{\beta^-} \ \& \ x \in U_{\beta,s}\},$$

$$\tau(\alpha, x, s) = \{\hat{e}_\beta : \beta \subseteq \alpha \ \& \ \hat{e}_\beta > \hat{e}_{\beta^-} \ \& \ x \in \widehat{V}_{\beta,s}\}.$$

(ii) The *final $\alpha$-state of $x$* is $\nu(\alpha, x) = \langle \alpha, \sigma(\alpha, x), \tau(\alpha, x) \rangle$ where $\sigma(\alpha, x) = \lim_s \sigma(\alpha, x, s)$ and $\tau(\alpha, x) = \lim_s \tau(\alpha, x, s)$.

**4.4. Templates as a Guide.** For $\alpha \in T$ we refer to the various objects, associated with $\alpha$, $\mathcal{M}_\alpha$, $\mathcal{N}_\alpha$, and $k_\alpha$ as *templates*[1] because from this information alone $\alpha$ can enumerate the appropriate sets (during those stages when $\alpha$ is accessible, namely $\alpha \subset f_s$). If $\alpha$ is on the true path then these sets will be correct.

The crucial point of the template-tree method is that each node $\alpha \in T$ at a given level works completely independently, acting only when it is accessible, and using its templates as a guide to its action.

To *relativize* the templates to $A$ ($\overline{A}$) means to restrict to only those states in $A$ ($\overline{A}$) respectively, and similarly for $B$ and $\overline{B}$. For example, $\mathcal{M}_\alpha^{\overline{A}}$ and $\mathcal{N}_\alpha^{\overline{A}}$ refer to the well-visited and nonwell-resided $\alpha$-states of $\overline{A}$, respectively.

**4.5. Skeletons and Enumerations.**

Definition 4.5. An array of c.e. sets $\{U_n\}_{n\in\omega}$ is a *skeleton* (*basis* ) (for $\mathcal{E}$) if

$$(\forall e)(\exists n)[U_n =^* W_e].$$

Definition 4.6. (i) Given tree $T$ with true path $f \in [T]$, an *enumeration* $\mathbb{E}$ for $T$ is a simultaneous computable enumeration of c.e. sets $U_\alpha$, $V_\alpha$, and $\widehat{U}_\alpha$, $\widehat{V}_\alpha$, for $\alpha \in T$, such that $\{U_\alpha\}_{\alpha \subset f}$ and $\{V_\alpha\}_{\alpha \subset f}$ are both skeletons.

(ii) Let $\rho = f(0)$ and let $A = U_\rho$ and $B = \widehat{U}_\rho$ (the first sets to be matched as in the Automorphism Theorem, Theorem 6.5).

(iii)

$$^{\mathbb{E}}\mathcal{M}_\alpha = \{\nu : (\exists^\infty x)(\exists s)[\nu(\alpha, x, s) = \nu \ \& \ x \in Y_{\alpha,s}]\},$$

and $^{\mathbb{E}}\widehat{\mathcal{M}}_\alpha$ is defined similarly on $\widehat{\omega}$-side.

[1]The Merriam-Webster Collegiate Dictionary defines "template" as a "gauge, pattern, or mold used as a guide to the form of a piece being made; a molecule (as of DNA) that serves as a pattern for the generation of another macromolecule (as messenger RNA)." We have carefully chosen the this word to explain our concept after exploring many others with similar meaning, such as "blueprint." This conveys an important intuition into the entire automorphism machinery.

(iv) An enumeration $\mathbb{E}'$ is an *extension* of an enumeration $\mathbb{E}$ if $S \subseteq S'$ for each set $S = U_\alpha, V_\alpha, \widehat{U}_\alpha, \widehat{V}_\alpha$ of $\mathbb{E}$ and the corresponding set $S' = U'_\alpha, V'_\alpha, \widehat{U}'_\alpha, \widehat{V}'_\alpha$ of $\mathbb{E}'$.

There is a deliberate ambiguity in the templates. For example, the template $\mathcal{M}_\alpha$ is used to represent an abstract blueprint which $\alpha$ uses to build the flow of elements into $Y_\alpha$ and it is also used to denote the actual flow,

$$^{\mathbb{E}}\mathcal{M}_\alpha = \{\nu : (\exists^\infty x)(\exists s)[\nu(\alpha, x, s) = \nu \ \ \& \ \ x \in Y_{\alpha,s}]\},$$

For $\alpha \subset f$ these will be the same. Strictly speaking, the templates have the former meaning and we must attach the left hand superscript $\mathbb{E}$ to denote the latter, but we will omit this superscript when there is no danger of confusion and write merely $\mathcal{M}_\alpha$ to denote the actual flow when the intended $\mathbb{E}$ is clear.

Definition 4.7. Given an enumeration $\mathbb{E}$ with true path $f$ define for each set of templates, $\mathcal{M}_\alpha, \mathcal{K}_\alpha, \mathcal{N}_\alpha$, $\alpha \in T$, the "join" along the true path $f$,

$$\mathcal{S}_f = \bigcup \{\mathcal{S}_\alpha : \alpha \subset f\} \quad \text{and } \widehat{\mathcal{S}}_f = \bigcup \{\widehat{\mathcal{S}}_\alpha : \alpha \subset f\}. \tag{16}$$

where $\mathcal{S}_\alpha$ denotes $\mathcal{M}_\alpha, \mathcal{K}_\alpha, \mathcal{N}_\alpha$, or other $\alpha$-templates.

Convention 4.8. For fixed $\mathbb{E}$ and $f$ we often drop the subscript $f$ from $\mathcal{S}_f$ in Definition 4.7 and simply write $\mathcal{S}$.

Note that $\mathcal{S}$ is not strictly a template but rather an infinite set of templates $\mathcal{S}_\alpha$, but it enables us to simplify the statements of conditions because

$$\mathcal{S} = \widehat{\mathcal{S}} \iff (\forall \alpha \subset f)[\mathcal{S}_\alpha = \widehat{\mathcal{S}}_\alpha].^2$$

Definition 4.9. For a fixed enumeration $\mathbb{E}$ define two other sets,

$$\mathcal{G}^A_\alpha = \{\nu : (\exists^\infty x)(\exists s)[x \in A_{s+1} - A_s \ \ \& \ \ \nu = \nu(\alpha, x, s)]\},$$

$$\widehat{\mathcal{G}}^B_\alpha = \{\nu : (\exists^\infty \hat{x})(\exists s)[x \in B_{s+1} - B_s \ \ \& \ \ \nu = \nu(\alpha, \hat{x}, s)]\}.$$

Think of $\mathcal{G}^A$ as a "gatekeeper set" consisting of nodes $\nu \in \mathcal{M}^{\overline{A}}$ from which infinitely many elements $x$ leave to enter $A$.

## 5. The New Extension Theorem

The main tool for almost our results on automorphisms is the following.

Theorem 5.1 (New Extension Theorem (N.E.T.)). *If an enumeration $\mathbb{E}$ satisfies both:*

(T1) $\mathcal{K}^{\overline{A}} = \widehat{\mathcal{K}}^{\overline{B}}$, *[Static equality of states of $\overline{A}/\overline{B}$], and*

(T2) $\mathcal{G}^A = \widehat{\mathcal{G}}^B$, *[Covering],*

*then $A \cong B$ by an enumeration $\mathbb{F}$ which extends $\mathbb{E}$.*

[2] The direction from right to left holds because for every $\beta <_{\mathrm{L}} f$ the set $U_\beta$ finite, and for every $\beta >_{\mathrm{L}} f$ we discard each $\beta$-set infinitely often. Therefore, each $\beta$-set $S_\beta$, $\beta \not\subset f$, may be taken to be finite, where $S_\beta$ is a variable ranging over $U_\beta, V_\beta, \widehat{U}_\beta, \widehat{V}_\beta$. Hence $\mathcal{M}_\beta$ and all its subset templates are empty for $\beta \not\subset f$. Therefore, the equation (16) could have been defined with $(\forall \alpha \in T)$ in place of $(\forall \alpha \subset f)$.

REMARK 5.2. (i) (T1) implies that $\mathcal{L}(A) \cong \mathcal{L}(B)$, and hence ensures the equality between the static (well-resided) states of $\overline{A}$ and $\overline{B}$.

(ii) (T2) asserts intuitively that $A$ (exactly) *covers* $B$ in the sense that if infinitely many elements enter $B$ from a state $\widehat{\nu}$ of $\overline{B}$ then infinitely many must enter $A$ from the same state $\nu$ and conversely.[3]

(iii) Of the two conditions, (T2), is much more recognizable, and of course, both are necessary, but satisfying (T1) is in general much more delicate and requires the more ingenious construction.

(iv) We normally satisfy (T1) above by satisfying the stronger *dynamic* condition,

$$\text{(T1)}' \quad (\mathcal{M}^{\overline{A}},\, \mathcal{N}^{\overline{A}}) = (\widehat{\mathcal{M}}^{\overline{B}},\, \widehat{\mathcal{N}}^{\overline{B}}). \qquad \text{[Dynamic matching of } \overline{A} \text{ and } \overline{B}]$$

(v) In many cases, for example if $A$ and $B$ are promptly simple or maximal, we satisfy (T2) by satisfying both (T1)$'$ and

$$\text{(T2)}' \quad \mathcal{M}^{\overline{A}} = \mathcal{G}^{A} \quad \& \quad \widehat{\mathcal{M}}^{\overline{B}} = \widehat{\mathcal{G}}^{B}. \qquad \text{[Autocover]}$$

(vi) For simplicity of exposition, we have deliberately not stated the N.E.T. in its strongest form. See the General Extension Theorem, Theorem 6.9, (G.E.T.) which has the same proof.

Note that Condition (T1)$'$ implies (T1) because $\mathcal{K}^{\overline{A}} = \mathcal{M}^{\overline{A}} - \mathcal{N}^{\overline{A}}$ as in (12). However, (T1)$'$ is much stronger than (T1) because it asserts that the correspondence is not merely a static one at the end of the construction, but a *dynamic* one describing corresponding states *during* the construction.

Since we always have $\mathcal{G}^{A} \subset \mathcal{M}^{\overline{A}}$ and likewise for $B$, condition (T2)$'$ asserts that $\mathcal{G}^{A}$ is as large as possible, which we call the *Autocover* case because $A$ (or more precisely $\mathcal{G}^{A}$) is dynamically covering its own complement $\overline{A}$ (or more precisely $\mathcal{M}^{\overline{A}}$). In the presence of (T1)$'$, note that (T2)$'$ easily implies (T2) by transitivity of equality. Hence,

$$[(T1)' \;\&\; (T2)'] \quad \Longrightarrow [(T1) \;\&\; (T2)] \quad \Longrightarrow A \simeq B. \tag{17}$$

In fact, most automorphisms, both effective and $\Delta^0_3$, have been built to satisfy (T1)$'$ and (T2)$'$ rather than merely (T1) and (T2). One of the most tempting pitfalls in the subject, leading to several false conjectures, has been the following fact.

$$(T1) \;\&\; (T2)' \quad \not\Longrightarrow \quad A \simeq B. \tag{18}$$

It is very tempting, for example, to assert that if two promptly simple sets, $A$ and $B$, satisfy $\mathcal{L}(A) \cong \mathcal{L}(B)$ then $A \simeq B$, but that is false. For a counterexample see Harrington and Soare [1998, p. 123] where $B$ is low, $A$ is low$_2$ with $\mathcal{L}(A) \cong \mathcal{L}(B) \cong \mathcal{E}$, and $A$ and $B$ are promptly simple, but $A \not\simeq B$.

[3]In the earlier effective Extension Theorem machinery this was stated in a form that for such $\nu$ in $\overline{B}$ there must be $\nu' \geq \nu$ meaning that $\sigma' \supseteq \sigma$ and $\tau' \subseteq \tau$, but in the Harrington-Soare $\Delta^0_3$-machinery we have equality of well visited states on $\overline{A}$ and $\overline{B}$ which makes the covering hypothesis (T2) much simpler than the old set of hypotheses in Soare [1987, 352–353].

**5.1. Duality.** One of the beautiful things about studying automorphisms of $\mathcal{E}$, especially with the $\Delta^0_3$-method, is the duality between the $\omega$ and $\widehat{\omega}$ sides. The hypotheses (T1) and (T2) ensure that $\mathcal{K}^{\overline{A}} = \widehat{\mathcal{K}}^{\overline{B}}$ and $\mathcal{G}^A = \widehat{\mathcal{G}}^B$. These properties ensure that for any template $\mathcal{S}$ defined from them we have,

$$\mathcal{S} = \widehat{\mathcal{S}} \qquad \text{[Duality]} \tag{19}$$

where the "dual" $\widehat{S}$ is obtained from $\mathcal{S}$ by replacing $\omega$, $U_\alpha$, and $\widehat{V}_\alpha$ by $\widehat{\omega}$, $\widehat{U}_\alpha$, and $V_\alpha$, respectively.

In the following we will define various templates, such as $\mathcal{W}_\alpha$, $\mathcal{C}_\alpha$, and $\mathcal{D}_\alpha$. We will show that various properties on $A$, such as noncomputability, promptly simplicity, promptness, simplicity, imply various corresponding properties for these templates, and will immediately conclude that the dual template has the same properties. This is exactly what is needed for the coding in Theorems 7.1, 7.2, 7.5, and others.

**5.2. Advantages of the New Extension Theorem.** There are a number of advantages of the New Extension Theorem over the old Extension Theorem in Soare [1987, p. 352].

1. Compared to the old Extension Theorem, this New Extension Theorem is more powerful and versatile, producing $\Delta^0_3$-automorphisms. These apply, for example, to Theorem 7.1 that any noncomputable c.e. set is automorphic to a high one, which effective automorphisms do not. The fact that the New Extension Theorem can be done on a tree while the old one cannot makes it much more applicable.
2. The N.E.T. can be applied wherever the old Extension Theorem applies, except that now we produce a $\Delta^0_3$-automorphism instead of an effective one. However, this slightly weaker conclusion is usually of little importance compared to extra versatility.
3. The two new hypotheses, while similar to the old ones, are much more compact and simple to state. This comes partly from the tree notation, and partly from the fact that the Harrington-Soare method ensures equality of states in $\mathcal{M}$ so we do not have to use inequalities to state the "cover" and "co-cover" properties as in Soare [1987, p. 352].

The New Extension Theorem has the following advantages over the plain Harrington-Soare $\Delta^0_3$-automorphism method [1996c] on which it is based.

1. The Harrington-Soare $\Delta^0_3$-method was designed for the applications in that paper and a few others, so the most general statement is the Automorphism Theorem [1996c, p. 633], which refers to the details of the construction there and says roughly that any variation of the construction which satisfies four conditions, stated in terms of that construction, produces an automorphism of $\mathcal{E}$. This makes it less portable than N.E.T. to another user or even to the authors themselves who need N.E.T. in their proof (Harrington-Soare [ta]) that $\overline{\mathbf{L}_1}$ is not invariant. To apply it one must get into the details of the construction.
2. Other users and the authors themselves who wished to apply a modified or extended version of the Harrington-Soare $\Delta^0_3$-method had to write things like, "Go to page 631 of Harrington-Soare, change this to that, and then modify the lemmas as follows." Now the reference can just be the statement

of the N.E.T., a statement *external* to the the steps of the construction, and the reader need not go into the construction at all.

Of course, the N.E.T. is proved with the Harrington-Soare $\Delta^0_3$-method, so the proof does not require a completely new method, but now we do not have to reapply the latter to build the automorphism taking $A$ to $B$. However, we often still use the $\Delta^0_3$-method to satisfy (T1), namely guaranteeing the isomorphism from $\overline{A}$ to $\overline{B}$.

## 6. Type 1 Automorphisms: Where We Are Given Both A And B

We now begin a number of applications of the New Extension Theorem to give simple proofs of theorems in the literature. These are only proof sketches, but are nearly full proofs. In these proofs the ideas of the proof may be the same as in the original author's papers, but the presentations are recast in the more general N.E.T. framework for greater perspicuity, and from the N.E.T. we can give shorter, cleaner proofs than from the original Soare Extension Theorem which all the authors used.

**6.1. Promptly Simple Low Sets.** Let's begin to translate normal properties of computability theory into the abstract template properties used in the N.E.T.

Lemma 6.1 (Maass, 1981). *If we are given an enumeration $\mathbb{E}$ and a promptly simple set $A$ then we can find an new enumeration $\mathbb{F}$ which speeds up the enumeration of $A$, leaves the enumeration of all other sets alone, and satisfies (T2)$'$ : $\mathcal{G}^A = \mathcal{M}^{\overline{A}}$.*

Proof. Let $\{\widetilde{A}_s : s \in \omega\}$ be the enumeration of $A$ in $\mathbb{E}$. For each $\alpha \in \mathcal{M}^{\overline{A}}$ and each $\alpha$-state $\nu$ we will define a c.e. set $Z_\nu$ which has Kleene index $W_{h(\nu)}$ for a computable function $h$. Whenever a new element $x$ appears in state $\nu$ at stage $s$ of enumeration $\mathbb{E}$, we put $x$ into $Z_\nu$ at stage $s$, wait for the first $t$ such that $x \in W_{h(\nu),t}$ and compute $\widetilde{A}_{p(t)}$. Let $u$ be the maximum of these $p(t)$ over all such $\nu$ at $s$, and define $A_s = \widetilde{A}_{p(u)}$. □

Theorem 6.2 (Soare, 1982). *If $A$ is low (or even if $\overline{A}$ is merely semilow), then $\mathcal{L}^*(A) \cong \mathcal{E}^*$ by an isomorphism effective on indices.*

This theorem and other applications of lowness use the Lowness Lemma 6.3 and (T3) below whose explanation requires a bit more notation. In the full Harrington-Soare $\Delta^0_3$ automorphism method [1996c, §2], the set $\mathcal{N}_\alpha$ of states being emptied out was split into the disjoint union $\mathcal{N}_\alpha = \mathcal{R}_\alpha \sqcup \mathcal{B}_\alpha$, where the nodes in $\mathcal{R}_\alpha$ are being emptied by RED and those in $\mathcal{B}_\alpha$ by BLUE . We need $\mathcal{R}_\alpha = \widehat{\mathcal{B}_\alpha}$ so that if RED empties state $\nu \in \mathcal{R}_\alpha$ then BLUE empties the same state in $\widehat{\mathcal{B}}_\alpha$ on the $\widehat{\omega}$-side. The heart of the whole method is to prove $\mathcal{R}$-consistency, namely

$$(\forall \nu_0 \in \mathcal{R}_\alpha)(\exists \nu_1)[\nu_0 <_R \nu_1 \ \ \& \ \ \nu_1 \in \mathcal{M}_\alpha], \tag{20}$$

If this holds, then by (19) (duality),

$$(\forall \hat{\nu}_0 \in \widehat{\mathcal{B}}_\alpha)(\exists \hat{\nu}_1)[\hat{\nu}_0 <_B \hat{\nu}_1 \ \ \& \ \ \hat{\nu}_1 \in \widehat{\mathcal{M}_\alpha}]. \tag{21}$$

Hence, if BLUE must empty out a state $\nu_0 \in \widehat{\mathcal{M}}_\alpha$ there is a state $\nu_1 \in \widehat{\mathcal{M}}_\alpha$ into which he can move the elements.

To prove Theorem 6.2 we must essentially match $\overline{A}$ with $\overline{B} = \widehat{\omega}$ so *no* elements of $\overline{B}_s$ ever leave $\overline{B}$ even if elements of $\overline{A}_s$ leave to enter $A$. Relativizing all the previous templates to $\overline{A}$, lowness allows us to prove the next lemma.

LEMMA 6.3 (Lowness Lemma). *If $A$ is low then the enumeration $\mathbb{E}$ can be arranged so that we have the following.*

$$(T3) \qquad (\forall \nu_0 \in \mathcal{R}_\alpha^{\overline{A}})(\exists \nu_1)[\nu_0 <_R \nu_1 \ \& \ \nu_1 \in \mathcal{M}_\alpha^{\overline{A}}].$$

This means that if a state $\nu_0$ of $\overline{A}$ is emptied by RED, he cannot empty *all* its elements into $A$, but must empty infinitely many into some other state $\nu_1$ of $\overline{A}$. Hence, on the $\widehat{\omega}$-side BLUE can empty the state $\nu_0$ of $\overline{B}$ into state $\nu_1$ still in $\overline{B}$.

PROOF. (Sketch) For each state $\nu \in \mathcal{M}^{\overline{A}}$ choose a marker $\Gamma$. Using the Robinson lowness trick in Soare [1987, pp. 224-228] ask whether $Q : Z \cap \overline{A} \neq \emptyset$ where $Z$ is the set of future positions of $\Gamma$. Move $\Gamma$ to some $x$ in $\nu$ only when one exists and the $0'$-oracle says yes to Q. Hold $x$ against any BLUE changes so if $\nu_0$ is emptied by RED , he must move some such $x$ into a state $\nu_1$ of $\overline{A}$. Repeat with infinitely many markers to get infinitely many such $x$. □

Since this is a result on $\mathcal{L}^*(A)$ rather than on automorphisms of $A$, Theorem 6.2 is not affected by the N.E.T., although the $\Delta_3^0$-method gives a more perspicuous proof for the noneffective case.

THEOREM 6.4 (Maass, 1981). *If $A$ and $B$ are low and promptly simple then $A \cong B$.*

PROOF. By Lemma 6.1 we have (T2)$'$. The proof of Theorem 6.2 guarantees (T1)$'$. Hence, by N.E.T. we have $A \cong B$. □

### 6.2. Automorphisms of Maximal Sets.

THEOREM 6.5 (Soare, 1974). *If $A$ and $B$ are maximal sets then $A \simeq B$.*

PROOF. We will modify the fixed enumeration $\{U_n\}$ to get a skeleton $\{U_n'\}$, and then measure states $\nu$ with respect to the latter. Since $A$ is maximal we have for each $W$ either $W \subset^* A$ or $W \supset^* \overline{A}$, namely that $\overline{A}$ obeys a *zero-one law.*

First, we can arrange that the enumeration of $U_n'$ is *order-preserving* in the sense that if an element, while in $\overline{A}$, appears in both $U_0'$ and $U_1'$ then it appears in $U_0'$ *first*, namely that $({U_1}' \searrow {U_0}') \setminus A = \emptyset$.

Second, if an element $x \in \overline{A}_s$ in state $\nu_1$ enters $U_n$, and if state $\nu_2$ ($\nu_2'$) corresponds to $\nu_1$ together with $U_n$ ($U_n'$), then we withhold $x$ from $U_n'$ either forever, or until some new element $y$ in state $\nu_2$ enters $A$ and put $y$ into $U_n'$ just before allowing $y$ to enter $A$. This guarantees (T2)$'$ in a very strong way because the element $x$ is not even allowed to enter state $\nu_2'$ while in $\overline{A}$ until a new witness in state $\nu_2$ enters $A$. This construction guarantees (T1)$'$ and (T2)$'$, and hence the automorphism by applying the N.E.T. □

REMARK 6.6. Although maximal sets may be either prompt or tardy, the previous proof shows that *maximal sets are always promptly simple on a skeleton*, and hence satisfy (T2)$'$, the crucial property for automorphisms.

QUESTION 1. All the known examples automorphisms of type 1 (where $A$ and $B$ are given and satisfy some property) require the Autocover (a kind of prompt simplicity) condition (T2)$'$ either directly or indirectly to guarantee (T2). Can one find examples of type 1 automorphisms which do not satisfy (T2)$'$?

**6.3. Automorphisms of Hyperhypersimple Sets.** Lachlan (Theorem 2.5) proved that a set $A$ is hh-simple iff $\mathcal{L}(A)$ is a Boolean algebra. Maass realized that the technique used for the maximal sets can be applied to hh-simple sets.

Theorem 6.7 (Maass, 1984). *If $A$ and $B$ are both hyperhypersimple, and if $\mathcal{L}(A) \cong_{\Delta^0_3} \mathcal{L}(B)$, then $A \simeq B$.*

Proof. There are $\Delta^0_3$ sequences $X_n$ and $Y_n$ such that the correspondence $X_n \mapsto Y_n$ induces the isomorphism $\mathcal{L}(A) \cong_{\Delta^0_3} \mathcal{L}(B)$, and the $X_n$ are disjoint on $\overline{A}$. The key point is that for each $U$ and $X$, $U$ satisfies the 0-1 law on $X \cap \overline{A}$. Hence, the same proof as in the maximal set case guarantees (T2)$'$. Namely, when an $x$ in state $\nu_1$ in $\overline{A}_s$ enters $U$, withhold it from state $\nu_2'$ until an element $y$ in state $\nu_2$ enters $A$. If $U \cap X \cap \overline{A} =^* \emptyset$ then the finite restraint will not matter. If $U \supseteq (X \cap \overline{A})$ then every such element will eventually become unrestrained. $\square$

While the preceding theorem shows that for hh-simple sets the hypothesis $\mathcal{L}(A) \cong_{\Delta^0_3} \mathcal{L}(B)$ is sufficient for the automorphism, surprisingly it is also *necessary.*

Theorem 6.8 (Cholak-Harrington, ta). *If $A$ and $B$ are hh-simple and $A \simeq B$ then $\mathcal{L}(A) \cong_{\Delta^0_3} \mathcal{L}(B)$*

This reduces the problem of classifying the automorphism types of hh-simple sets to that of classifying their $\mathcal{L}(A)$ which Lachlan [1968] has done. (See Soare [1987, p. 203].)

**6.4. Hemi-Maximal Sets.** Downey and Stob [1992, p. 237] defined a set $H$ to be *hemi-maximal* if there are a maximal set $M$ and disjoint noncomputable c.e. sets $A_0$ and $A_1$ such that $H = A_0$, and $M = A_0 \sqcup A_1$ is a Friedberg splitting. Let $\sqcup$ denote disjoint union. A disjoint splitting $A = A_0 \sqcup A_1$ is a *Friedberg splitting* if $W - A$ not c.e. implies $W - A_i$ not c.e., $i = 0, 1$. It is easy to see (Downey and Stob [1992, p. 239]) that any nontrivial splitting of a maximal set is a Friedberg splitting.

Downey and Stob showed that the hemi-maximal sets form an orbit, namely any two are automorphic. To deduce their theorem in the present context consider the following theorem whose proof is nearly identical to that of the N.E.T.

Theorem 6.9 (General Extension Theorem (G.E.T.)). *Suppose that $A = \sqcup_{1 \le i \le n} A_i$ and $B = \sqcup_{1 \le i \le n} B_i$. If an enumeration $\mathbb{E}$ satisfies both:*

(T1) $\mathcal{K}^{\overline{A}} = \widehat{\mathcal{K}}^{\overline{B}}$, *[Static equality of states of $\overline{A}/\overline{B}$], and*

(T2)$_i$ $\mathcal{G}^{A_i} = \widehat{\mathcal{G}}^{B_i}$, *for all $i \le n$ [Covering],*

*then there is an enumeration $\mathbb{F}$ which extends $\mathbb{E}$ which witnesses $A_i \simeq B_i$, for all $i \le n$.*

The proof is exactly the same as for the N.E.T. except that elements entering $A$ immediately enter a unique set $A_i$ and play against those elements entering the corresponding set $B_i$. The N.E.T. Theorem 5.1 and its proof bear the same relation to G.E.T. as the original Friedberg-Muchnik Theorem bears to the trivial extension that there are infinitely many c.e. sets of incomparable Turing degree (see Soare [1987, p. 120]).

Theorem 6.10 (Downey Stob, 1992). *The hemi-maximal sets form an orbit.*

Proof. Let $A$ and $B$ be maximal and $A = A_0 \sqcup A_1$ and $B = B_0 \sqcup B_1$. Now by the proof of the Maximal Set Theorem 6.5, there is an enumeration $\mathbb{E}$ satisfying (T1)$'$ and (T2)$'$. By exactly the same method as in the Maximal Set Theorem 6.5, we can guarantee $(\text{T2})'_i : \mathcal{G}^{A_i} = \widehat{\mathcal{G}}^{B_i}$, for all $i \leq n$, from which the theorem immediately follows by the General Extension Theorem 6.9.

Namely, as in the Maximal Set Theorem 6.5 if an element $x \in \overline{A}_s$ in state $\nu_1$ enters $U_n$, and if state $\nu_2$ ($\nu_2'$) corresponds to $\nu_1$ together with $U_n$ ($U_n'$), then we withhold $x$ from $U_n'$ either forever, or until some new element $y$ in state $\nu_2$ enters $A_0$ *and* a new element $z$ in state $\nu_2$ enters $A_1$. If $U \supseteq \overline{A}$, infinitely many such must enter $A$, and by the Friedberg splitting of $A$ into $A_0$ and $A_1$, infinitely many such $x$ must enter *each* of $A_0$ and $A_1$. ☐

Exactly the same method may be used to show $A_0 \simeq B_0$ for any two sets $A$ and $B$ which are automorphic using the 0-1 property as if the maximal set case and and which are then Friedberg into $A_0 \sqcup A_1$ and $B_0 \sqcup B_1$. This includes hemi-maximal sets, Friedberg splittings of hh-simple sets with $\mathcal{L}^*(A) \cong_{\Delta_3} \mathcal{L}^*)B)$, Herrmann sets, and others.

Question 2. In Theorem 6.10 we have seen one case where Friedberg splittings of automorphic sets are also automorphic. Which orbits have the property that for any two sets $A$ and $B$ in the orbit and any Friedberg splittings $A = A_0 \sqcup A_1$ and $B = B_0 \sqcup B_1$ we have $A_0 \simeq B_0$?

Question 3. Under what conditions on $A$ *alone* is it true that any Friedberg splitting $A = A_0 \sqcup A_1$ satisfies $A_0 \simeq A_1$?

For example, if $A$ is promptly simple and $A = A_0 \sqcup A_1$ is a Friedberg splitting, then is $A_0 \simeq A_1$? At first one might conjecture yes because: (1) it works for maximal sets, and both maximal and promptly simple sets satisfy (T2)$'$ and hence (T2) for N.E.T.; (2) if the splitting is a prompt splitting in the sense of Downey-Stob [1993a, p. 181] (as Friedberg's original splitting theorem was prompt) then the conjecture is clearly true. However, the Friedberg splitting condition only guarantees a flow from $\overline{A}$ to $A_i$ and not a prompt one, so $A_0$ may never get the elements from $\overline{A}$ *while* they are in a desirable state $\nu$, and by the time they arrive later, they may have changed state. Some of these questions and others on splittings were raised in Downey-Stob [1992], [1993], [1993b], who showed that all Friedberg splittings of creative sets are automorphic.

Russell Miller turned these doubts into a theorem as follows. First he proved that there is a promptly simple set $A$ and a Friedberg splitting $A = A_0 \sqcup A_1$ such that $A_0$ is tardy and $A_1$ is prompt. Now by Theorem 7.5, $A_1$ is automorphic to a complete set, but we need something a little stronger to conclude that $A_0$ is not. Miller invented a new $\mathcal{E}$-definable property for splittings like the $Q(A)$ property of Theorem 3.5 which guarantees that $A_0$ is $\mathcal{E}$*-definably* tardy, and hence that every set in its orbit is incomplete.

Theorem 6.11 (Russell Miller, 2000). *(i) There is an $\mathcal{E}$-definable property $R(A_0, A_1)$ which implies that $A_0 \sqcup A_1$ is a Friedberg splitting of $A = A_0 \cup A_1$ and implies that $A_0$ is tardy.*

*(ii) There exists a c.e. set $A$ with a Friedberg splitting $A = A_0 \sqcup A_1$ such that all of the following hold: $A$ is promptly simple of high degree; $A_1$ has prompt degree; and $R(A_0, A_1)$ holds.*

It follows that $A_0$ and $A_1$ cannot be automorphic.

QUESTION 4. If we cannot answer the preceding questions positively with automorphisms, what kind of $\mathcal{E}$-definable properties can we exhibit like Russell Miller's to refute their existence? What other new definable properties or variations on existing ones can we find to refute the existence of automorphisms?

**6.5. Low Simple Sets.** The most general question remaining for type 1 automorphisms is this.

QUESTION 5. Classify properties $P(X)$ (like those above) such that $P(A)$ and $P(B)$ guarantee that $A$ and $B$ are automorphic.

Since this is an immense task, we restrict it in this section just to simple sets and even more to low ones.

QUESTION 6. If $A$ and $B$ are low simple sets, classify conditions $P$ which guarantee that they are automorphic.

The reason for starting with low sets $A$ and $B$ is that by Theorem 6.2 we have $\mathcal{L}^*(A) \cong_{eff} \mathcal{L}^*(B)$, and by (T3) we have great control over which states of $\overline{A}$ will empty out. Now if $A$ and $B$ are promptly simple then by Theorem 6.4 they are automorphic, so we are now looking at the nonautocover case. We still need to satisfy (T2). Hence, the key is to find a new way of approaching the following question.

QUESTION 7. If $A$ and $B$ are low, simple, and nonpromptly simple, how do we control the enumeration to achieve (T2) in the absence of (T2)$'$?

Namely, we begin by allowing some element $y$ of $\overline{B}$ to enter a state $\nu$, but since we are in a type 1 automorphism construction where the opponent controls both $A$ and $B$, we must expect that he will put $y$ into $B$ immediately if it suits him. Hence, we must have anticipated this move by previously forcing some $x$ of $\overline{A}$ in state $\nu$ into $A$. How do we select which states, and what property of $A$ (analogous to maximal or promptly simple) can be used to force such a move by $x$? (Can $d$-simple or non-$d$-simple be used?) This is the next major front in the type 1 games, because all the known cases use (T2)$'$.

**6.6. Atomless r-Maximal Sets.**

DEFINITION 6.12. (i) A coinfinite set $A$ is *r-maximal* if there is no computable set $R$ such that both $R \cap \overline{A}$ and $\overline{R} \cap \overline{A}$ are infinite.

(ii) An $r$-maximal set $A$ is *atomic* if $A$ has a maximal superset, and *atomless* otherwise.

If $A$ and $B$ are atomic $r$-maximal then it follows that $\mathcal{L}(A) \cong \mathcal{L}(B)$ because they are major subsets in their respective maximal sets. (See Maass-Stob [1983]).

QUESTION 8. If $A$ and $B$ are atomic $r$-maximal sets, under what conditions is $A \simeq B$? The conclusion cannot always hold because the major subset may or may not be small. (See Soare [1987, p. 194].)

QUESTION 9. If $A$ and $B$ are atomless $r$-maximal sets, under what conditions is $A \simeq B$? Assume $\mathcal{L}(A) \cong \mathcal{L}(B)$ or even $\mathcal{L}(A) \cong_{\Delta^0_3} \mathcal{L}(B)$. What if $A$ and $B$ are also promptly simple?

Cholak-Nies [ta] have proved that there are infinitely many $r$-maximal sets $\{A_i\}$ such that $\mathcal{L}(A_i) \not\cong \mathcal{L}(A_j)$ for $i \neq j$. Hence, let us assume that $\mathcal{L}(A) \cong_{\Delta^0_3} \mathcal{L}(B)$. In the case of maximal or hh-simple sets we could achieve (T2)$'$ because we had promptness on a skeleton. Can we achieve something like that here, and if not, can we discard (T2)$'$ and achieve (T2) by a more delicate balancing of the flows into $A$ and $B$? The following may be an easy question, not yet examined.

Question 10. If $A$ and $B$ are promptly simple atomless $r$-maximal sets and $\mathcal{L}(A) \cong_{\Delta^0_3} \mathcal{L}(B)$, then under what conditions is $A \simeq B$?

Lempp, Nies, and Solomon [ta] have proved that there is an atomless $r$-maximal set $A$ such that the set $\{e : W_e \cup A =^* \omega\}$ is $\Sigma_3$-complete. This implies that the set $A$ has no uniformly c.e. (u.c.e) weak tower in the sense of Soare [1987, p. 196].

**6.7. Pseudo-Creative Sets.** A set $A$ is *pseudo-creative* if it is not creative and for every $W \subset \overline{A}$ there is a infinite set $V$ disjoint from $A \cup W$. These sets $A$ stand in contrast to the simple sets where most of our automorphism results lie. It is difficult to see how to get started. Classifying the automorphism type of a pseudo-creative set requires a different approach from that for simple sets because many c.e. sets do not intersect $A$. Also E. Herrmann has isolated a very interesting class of pseudo-creative sets which form an orbit, surprisingly for the same reason as the maximal sets. Herrmann considers the class of sets which are pseudo-creative, $r$-separable, and $\mathcal{D}$-maximal (now called *Herrmann sets*), and proves they form an orbit. See Cholak-Downey [ta] for definitions and a proof in dynamic form, whereas the Herrmann proof was originally in static form but used the Maximal Set Theorem 6.5 for its dynamic component.

In the maximal set case we had a 0-1 law in which every new set W satisfied either $W \subseteq^* A$ or $W \supseteq \overline{A}$ (not merely $W \supseteq^* \overline{A}$), and so we got prompt simplicity on a skeleton and (T2)$'$. The remarkable thing about the Cholak-Downey-Herrmann proof is that, modulo computable sets, one gets essentially the same thing for the Herrmann sets. Hence, the ideas of the maximal set theorem and Extension Theorem can be applied.

Question 11. The Cholak-Downey-Herrmann theorem gives just *one* orbit of pseudo-creative sets. How can we classify others? Herrmann looked at sets modulo computable sets. Can we mod out by other classes to study the pseudo-creative sets? Can we begin a direct approach to achieving (T1) and (T2)? For example, if $A$ and $B$ are pseudo-creative and low, then we have $\mathcal{L}(A) \cong \mathcal{L}(B)$ so we need merely achieve (T2). Can we achieve promptness at least on a skeleton?

If $A$ is simple then $A \times \omega$ is pseudo-creative and $r$-separable, so we can form many pseudo-creative sets.

## 7. Type 2 Automorphisms: Given Only $A$

This section resembles the preceding section, except that we are given only $A$ and are required to construct an automorphic copy $B$ with certain properties. The theorems in this section are not simple applications of the New Extension Theorem because the procedures to satisfy (T1), or more usually (T1)$'$, are quite complicated.

Nevertheless, the principles of the New Extension Theorem still apply, and we still need to meet (T2) somehow. Here it is usually easier to satisfy (T2), and we rarely require that the hypotheses guarantee (T2)$'$. We control the flow of elements into $B$, so it is safer to allow an element $y$ of $\overline{B}$ to enter a state $\nu$ of $\overline{B}$ because we control whether $y$ later enters $B$ and hence whether (T2) is satisfied. Suppose we wish to put more information into $B$ than $A$ contains, for example to prove either of the following theorems.

### 7.1. Mapping $A$ to Some $B$ Which Codes Information.

THEOREM 7.1 (Harrington-Soare, 1996c, and Cholak, 1995). *For every non-computable c.e. set $A$ there is a high c.e. set $B$ such that $A \simeq B$.*

THEOREM 7.2 (Harrington, see Harrington-Soare, Theorem 9.1). *For all c.e. sets $A$ and $C$ such that $\emptyset <_T A$ and $C <_T K$ there is a c.e. set $B \simeq_{\Delta^0_3} A$ such that $B \not\leq_T C$.*

We use the noncomputability of $A$ to show that there are certain "coding nodes" of $\overline{A}$ which will carry over to $\overline{B}$ by (19) (duality). Define

$$\text{(22)}\qquad \mathcal{W}_\alpha = \{\nu : \nu \in \mathcal{K}^{\overline{A}}_\alpha \;\;\&\;\; \text{RED has a winning strategy to move any } x \text{ in state } \nu \text{ into } A\}.$$

Picture all the $\alpha$ states as a giant chess board containing the finitely many $\alpha$-states arranged so RED can move from $\nu_1$ to $\nu_2$ by enumerating $x$ in a red set $U_\beta$ for some $\beta \subseteq \alpha$ and BLUE can move from $\nu_1$ by enumerating $x$ in a blue set $\widehat{V}_\beta$.

Now $\nu \in \mathcal{W}_\alpha$ means that RED can keep $x$ in $\overline{A}$ forever if he wishes (because $\nu \in \mathcal{K}_\alpha$), but by a sequence of voluntary moves by himself and forced moves by BLUE, RED can slowly move $x$ into $A$. A typical forced move for BLUE is this. Suppose we know that $U_\beta \subset \widehat{V}_\gamma$. Then RED moves $x$ into $U_\beta$ by a voluntary red move, and waits for BLUE to move $x$ into $\widehat{V}_\gamma$, which is now forced. An easy precise mathematical definition of $\mathcal{W}_\alpha$ from $\mathcal{M}_\alpha$ and the other parameters is given in Harrington-Soare [1996c, Definition 6.1].

Define $\nu <_R \nu'$ to hold if $\nu = \langle \alpha, \sigma, \tau \rangle$ and $\nu' = \langle \alpha, \sigma', \tau \rangle$ where $\sigma \subset \sigma'$. Define $\nu <_B \nu'$ to hold if $\nu = \langle \alpha, \sigma, \tau \rangle$ and $\nu' = \langle \alpha, \sigma, \tau' \rangle$ where $\tau \subset \tau'$. The idea is that on the $\omega$-side, any $x$ in state $\nu$ is a RED (BLUE) move away from entering state $\nu'$, because RED (BLUE) can enumerate $x$ in any set $U_\beta$ ($V_\beta$) to raise the $\sigma$ component of the state of $x$ from $\sigma$ to $\sigma'$ ($\tau$ to $\tau'$). (On the $\widehat{\omega}$-side it is the reverse.)

Define

$$\text{(23)}\qquad \mathcal{P}_\alpha = \{\nu : \nu \in \mathcal{M}_\alpha \;\;\&\;\; \neg(\exists \nu' \in \mathcal{M}_\alpha)[\nu <_B \nu']\}.$$

These nodes are *opponent maximal* in the sense that the opponent, BLUE, cannot move any $x$ on the $\omega$-side from $\nu$ to another state. The following *coding nodes* $\mathcal{C}_\alpha$ play a key role and are defined as follows in Harrington-Soare [1996c, Definition 6.2]. For $\alpha \in T$ define,

$$\text{(24)}\qquad \mathcal{C}_\alpha =_{\text{dfn}} \mathcal{W}_\alpha \cap \mathcal{P}_\alpha. \qquad\qquad \text{[Coding Nodes]}$$

This means that, in addition to $\mathcal{W}_\alpha$, we add the property $\mathcal{P}_\alpha$ that *BLUE* cannot move $x$ out of a state $\nu \in \mathcal{C}_\alpha$. Hence, RED can keep $x$ in $\nu$ and in $\overline{A}$ forever, because $\nu \in \mathcal{K}^{\overline{A}}_\alpha$, or at a later stage can begin to gradually send $x$ on a series of forced moves ending up in $A$, hence the word "coding nodes" as we will see. The first key fact about $\mathcal{C}_\alpha$ is that it is nonempty if $A$ is noncomputable.

THEOREM 7.3 (Harrington-Soare, 1996c, Lemma 6.4). *If $A$ is noncomputable and $\alpha \subset f$, the true path of $T$, then $\mathcal{C}_\alpha \neq \emptyset$.*

PROOF. (Sketch) Fix $\alpha \subset f$ and suppose $\mathcal{C}_\alpha = \emptyset$. Then almost every $x \in \overline{A}$ must eventually enter some $\alpha$-state $\nu \in \mathcal{K}_\alpha^{\overline{A}}$. Now RED has no winning strategy to force $x$ into $A$ from $\nu$, because $\nu \notin \mathcal{C}_\alpha$. It is easy to convert this (see Harrington-Soare [1996]) to the assertion that $\nu \notin \mathcal{W}_\alpha$. But the game is obviously determined. Hence, BLUE must have a winning strategy to keep $x$ *out* of $A$. Now since this applies to all $\nu \in \mathcal{K}_\alpha^{\overline{A}}$ BLUE can computably enumerate $\overline{A}$. Hence, $A$ is computable, contrary to hypothesis. □

Hence, $A$ noncomputable implies $\mathcal{C}_\alpha \neq \emptyset$ which, by (19) (duality), implies $\widehat{\mathcal{C}}_\alpha \neq \emptyset$. These nodes can be used for elements $y$ of $\overline{B}$ to rest in $\overline{B}$ forever, or to be forced by BLUE into $B$ at will. This is what is needed in proving Theorem 7.1 or 7.2.

**7.2. Avoiding an Upper Cone.** Theorem 7.2 uses the Sacks strategy of avoiding the downward cone as explained in Soare [1987, §4]. It is usually much easier to avoid an *upper* cone using the Sacks preservation strategy as in Soare [1987, p. 122], but here the following question is still open.

QUESTION 12. [**Avoiding an Upper Cone**]

$$(\forall A <_{\mathrm{T}} \emptyset')(\forall C >_{\mathrm{T}} \emptyset)(\exists B \not\geq_{\mathrm{T}} C)[A \simeq B]?$$

The point is that to avoid the upper cone we will have to put some restraint on the enumeration of $B$ as Sacks did. Doing so may be impossible if $A$ is complete as the properties CRE($A$) and T($A$) of §3 show, so we must require that $A$ be incomplete. The noncomputability of $A$ ensured that $\mathcal{C}_\alpha \neq \emptyset$ above. The key question here is to find the corresponding property for $A$ incomplete.

QUESTION 13. Use the hypothesis $A <_{\mathrm{T}} \emptyset'$ to get a $\mathcal{C}_\alpha$-style property $P$ on $A$ which will translate on the $B$-side to a property which will allow elements to be restrained from $B$ for the Sacks negative preservation method.

Namely, what does the incompleteness of $A$ say about the enumeration of $A$ which prevents elements from appearing too quickly? The only progress on these questions so far is by Russell Miller [2000], after considerable effort by senior people,

THEOREM 7.4 (R. Miller, 2000). *If $A$ is low and $C >_{\mathrm{T}} \emptyset$, then there is a $B \not\geq_{\mathrm{T}} C$ such that $A \simeq B$.*

The idea is that the Lowness Lemma 6.3 and (T3) above prevent states $\nu$ on $\overline{A}$ from being emptied out unexpectedly. This, in turn, would force states in $\overline{B}$ to be emptied too fast and would interfere with the Sacks' restraint to avoid the cone above $C$.

**7.3. Prompt Sets and Completeness.** If we want to map a set $A$ to a complete set, we need more than just the hypothesis that $\mathcal{C}_\alpha \neq \emptyset$, because by Theorem 3.5 there are noncomputable sets $A$ which are not automorphic to any complete set. The first major progress in the completeness direction was the result by Cholak, Downey and Stob [1992] that if a set $A$ is promptly simple then $A$ is automorphic to a complete set. Harrington-Soare [1996c] then improved this by proving the following theorem.

Theorem 7.5 (Harrington-Soare, 1996c, §10). *If $A$ is prompt (or even almost prompt [1996c, §11]) then there is a complete set $B$ such that $A \simeq B$.*

The key point in the proof is to define a set of prompt coding nodes. For $\alpha \in T$ define,

$$\mathcal{D}_\alpha =_{\mathrm{dfn}} \mathcal{K}_\alpha^{\overline{A}} \cap \mathcal{P}_\alpha \cap \mathcal{G}_\alpha^A. \tag{25}$$

The intuition is that if $x \in \nu \in \mathcal{D}_\alpha$ then RED can hold $x$ in $\overline{A}$ forever as before by the first two clauses, but now, whenever he likes, RED can enumerate $x$ *immediately* into $A$ and not through a series of moves as when we only have $x \in \nu \in \mathcal{W}_\alpha$.

Translated onto the $\widehat{\omega}$-side this means that we have a state $\nu$ of $\overline{B}$ with infinitely many permanent residents in $\overline{B}$ but such that we can move any temporary resident *immediately* into $B$ for coding.

Theorem 7.6. *(i) If $A$ is prompt then $(\forall \alpha \subset f)[\mathcal{D}_\alpha \neq \emptyset]$.*

*(ii)If the enumeration of $A$ can be arranged so that $(\forall \alpha \subset f)[\mathcal{D}_\alpha \neq \emptyset]$, then $A$ is automorphic to a complete set.*

Why are the prompt coding nodes necessary? If an element $x$ in state $\nu \in \widehat{\mathcal{C}}_\alpha$ enters $B$ *eventually* why is that not good enough to show that every noncomputable set is automorphic to a complete one?

The reason is that Theorem 7.6 only holds for $\alpha \subset f$ and we cannot tell during the construction whether $\alpha \subset f$. If we place a coding marker on an element $x$ in state $\nu'$ for $\nu' \notin \mathcal{D}_\alpha$ where $\alpha \subset f$, then when $x$ begins its journey toward $B$ it may never receive the next move by the opponent which it expects, and may be stuck forever outside of $B$. For a coding marker this is a fatal error. In the case of $x \in \nu \in \mathcal{D}_\alpha$ for $\alpha \subset f$, however, there are no opponent moves to wait for, since BLUE can immediately enumerate $x$ into $B$ whenever he pleases. This bears on the next key question.

Question 14. Find a set of necessary and sufficient conditions for a set $A$ to be automorphic to a complete set.

Previously, one looked at this problem from the point of view of two sufficient conditions: maximal (and its derivatives like hh-simple, and hemi-maximal) and promptness or almost promptness. One of the contributions of this paper is to point out that maximal sets are promptly simple on a skeleton. This suggests the following.

Question 15. Can one prove that a necessary and sufficient condition for a set $A$ to be automorphic to a complete set is that it be almost prompt on a skeleton?

This question is a kind of dual to Question 13. The evidence suggests at the moment that Question 15 is true for $\Delta_3^0$ automorphisms, but it is harder to get a handle on all automorphisms.

**7.4. Hitting a Cone.** Another theme in moving elements around is to try to hit a particular cone, say a downward cone. This is a result in that direction.

THEOREM 7.7 (Wald, 1999). *If $A$ is low (or even if $\overline{A}$ is semilow) and $C$ is promptly simple, then there is a $B \leq_{\mathrm{T}} C$ such that $A \simeq B$.*

PROOF. (Sketch) By the Lowness Lemma 6.3 states $\nu$ of $\overline{A}$ do not empty into $A$ so $B$ need not seek $C$-permission to empty corresponding states into $B$. Rather $B$ need only get $C$-permission move elements rapidly from $\overline{B}$ to $B$ to ensure $\widehat{\mathcal{G}}^B \supseteq \mathcal{G}^A$, something $C$ can easily permit by promptness. $\square$

THEOREM 7.8 (Wald, 1999). *If $A$ is low (or even if $\overline{A}$ is semilow) and promptly simple, and $C$ is promptly simple then there is a $B \equiv_{\mathrm{T}} C$ such that $A \simeq B$.*

PROOF. (Sketch) In addition to the former part, we now need to put elements $y_x$ into $B$ to code when some element $x$ enters $C$. To do this we need prompt coding nodes like $\nu \in \mathcal{D}_\alpha$ as in Theorem 7.5. $\square$

QUESTION 16. Does Theorem 7.8 hold if $A$ is merely prompt in place of being promptly simple?

At first it would seem so, but there are some delicate timing questions where the $C$-permitting does not get along with the approximation $f_s$ to the true path. Any attempt to move elements around is limited by the following.

THEOREM 7.9 (Downey-Harrington, ta). *There is a prompt low degree* $\mathbf{a}$ *and a tardy (nonprompt) degree* $\mathbf{b}$ *such that*

$$(\forall A \in \mathbf{a})(\forall B \leq_T \mathbf{b})[A \not\simeq B].$$

A corollary of Theorem 7.7 of Wald's thesis [Wald, 1999] is that we cannot improve Downey-Harrington Theorem 7.9 to "$(\forall A \leq_{\mathrm{T}} \mathbf{a})$." This also also follows by the existence of hemimaximals of arbitary low degree. The no fat orbit theorem in its full generality shows that no member of the $\mathrm{low}_n/\mathrm{high}_n$ hierarchy is defined by a single orbit, except for the $\mathrm{high}_1$ degrees. The next question was raised by Downey.

QUESTION 17. Which classes of degrees are definable by single orbits? We know the high degrees and the complete degree are such. Is there a nice characterization of any other?

Harrington and Cholak have shown that all double jump classes are definable.

**7.5. Prompt high orbits.** One of the most interesting threads in the subject has been that initiated by Martin's beautiful Theorem 2.3 that the degrees of maximal sets are the high degrees. This led to the conjecture that for every noncomputable $A$ and every high c.e. degree $\mathbf{d}$ there is a $B \in \mathbf{d}$ such that $A \simeq B$.

Maass, Shore, and Stob [1981] refuted this by producing an $\mathcal{E}$-definable property, $SP\overline{H}$ (possessing a certain splitting property and not being hh-simple), which distinguished between some prompt and nonprompt sets. Cholak [1995] proved a version of the conjecture by showing that the weaker conclusion "$\mathcal{L}(A) \cong \mathcal{L}(B)$" is true. It would be very interesting to know whether the original conjecture for automorphisms holds in the form where everything is promptly simple or prompt so the above barrier is removed. Note that we also need to add the hypothesis that $A$ is incomplete to avoid the properties $CRE(A)$ and $T(A)$ of §3.

QUESTION 18 (Prompt high orbit question). Given a promptly simple set $A <_{\mathrm{T}} \emptyset'$ and a high promptly simple set $D$, does there exist $B \equiv_T D$ such that $A \simeq B$?

The fact that $A$ and $D$ are both promptly simple allows us to achieve (T2)$'$, but not necessarily (T2) since we do not have (T1)$'$. The highness of $D$ allows us to empty states of $\overline{B}$ into $B$ to achieve (T1): $\mathcal{L}(A) \cong \mathcal{L}(B)$, as in Cholak, but is this enough? If we could achieve the dynamic property (T1)$'$ we would succeed, but this is not clear. This resembles the Harrington-Soare refutation of a similar conjecture described in the paragraph directly following (18) and immediately preceding §5.1 There we also had (T2)$'$ and (T1) only, but not (T1)$'$ and we failed to produce the automophism. This seems to be a subtle but crucial obstacle to building automorphisms and deserves a lot more attention.

### 7.6. Orbit Complete Classes.

Definition 7.10. For classes $\mathcal{S}$, $\mathcal{T} \subseteq \mathcal{E}$ we say that $\mathcal{S}$ is *orbit complete* in $\mathcal{T}$ if

$$(\forall X \in \mathcal{S})(\exists Y \in \mathcal{T})[X \simeq Y]. \tag{26}$$

Cholak [1998] answered a question of Herrmann by proving that the simple sets are orbit complete in the hypersimple sets. The key step here is to prove that

$$A \text{ simple } \implies \mathcal{W}_\alpha = \mathcal{M}^{\overline{A}} \text{ for } \alpha \subset f. \tag{27}$$

The proof is as in Theorem 7.3 for $\mathcal{C}_\alpha \neq \emptyset$ except that for *every* $\nu \in \mathcal{M}^{\overline{A}}$ RED must have a winning strategy to force any element $x \in \nu$ into $A$, for if not, then $B$ has a winning strategy to keep such $x$ in $\overline{A}$ and hence enumerate an infinite subset of $\overline{A}$. Wald [1999] answered a question of Jockusch by proving that the promptly simple sets are orbit complete in the effectively simple sets.

Question 19. For which other pairs of classes $\mathcal{S}$, $\mathcal{T} \subseteq \mathcal{E}$ is $\mathcal{S}$ *orbit complete* in $\mathcal{T}$?

## 8. Summary of Template Properties

The $\Delta^0_3$ paper by Harrington-Soare and the present paper highlight the desirability of translating standard computability-theoretic properties such as noncomputability, simplicity, promptness, and maximality into abstract properties about the templates. We now summarize some of these.

$(T1)$ $\mathcal{K}^{\overline{A}} = \widehat{\mathcal{K}}^{\overline{A}}$.

$(T2)$ $\mathcal{G}^{A} = \widehat{\mathcal{G}}^{B}$.

$(T1)'$ $(\mathcal{M}^{\overline{A}}, \mathcal{N}^{\overline{A}}) = (\widehat{\mathcal{M}}^{\overline{B}}, \widehat{\mathcal{N}}^{\overline{B}})$.

$(T2)'$ $\mathcal{M}^{\overline{A}} = \mathcal{G}^{A}$ & $\widehat{\mathcal{M}}^{\overline{B}} = \widehat{\mathcal{G}}^{B}$.

(T3) $(\forall \nu_0 \in \mathcal{R}^{\overline{A}}_\alpha)(\exists \nu_1)[\nu_0 <_R \nu_1 \ \& \ \nu_1 \in \mathcal{M}^{\overline{A}}_\alpha]$ [For $A$ low].

- **N.E.T. Theorem** $[(T1)' + (T2)'] \implies [(T1) + (T2)] \implies A \simeq B$.
- $A, B$ maximal (or hhs with $\mathcal{L}^*(A) \cong_{\Delta_3} \mathcal{L}^*(B)$ ) $\implies$ (T1)$'$ and (T2)$'$.
- $A, B$ Herrmann $\implies$ (T1)$'$ and (T2)$'$.
- $A_0$, $B_0$ hemimaximal $\implies$ (T1)$'$ and (T2)$'$.
- $A, B$ low $\implies$ (T1)$'$ and (T3).
- $A, B$ promptly simple $\implies$ (T2)$'$.

- $A$ simple $\implies \mathcal{W}_\alpha = \mathcal{M}_\alpha^{\overline{A}}$ (for (T2)$'$), where
  $\mathcal{W}_\alpha = \{\nu :$ RED has a winning strategy to move $x \in \nu$ into $A$ $\}$.
- $A$ noncomputable $\implies \mathcal{C}_\alpha \neq \emptyset$, $\alpha \subset f$, (toward (T2)$'$), where
  $\mathcal{C}_\alpha = \mathcal{K}_\alpha^{\overline{A}} \cap \mathcal{W}_\alpha \cap \mathcal{P}_\alpha$, and [Coding nodes]
  $\mathcal{P}_\alpha = \{\nu : \nu \in \mathcal{M}_\alpha \ \& \ \neg(\exists \nu' \in \mathcal{M}_\alpha)[\nu <_B \nu']\}$. [Blue maximal nodes]
- $A$ prompt $\implies \mathcal{D}_\alpha \neq \emptyset$, $\alpha \subset f$, toward (T2)$'$, where
  $\mathcal{D}_\alpha = \mathcal{K}_\alpha^{\overline{A}} \cap \mathcal{G}_\alpha^A \cap \mathcal{P}_\alpha$. [Fast coding nodes]
- $A \mapsto B \leq_{\mathrm{T}} C$ p.s. $\implies \widehat{\mathcal{G}}^B \supseteq \mathcal{G}^A$, toward (T2)$'$, where $A \mapsto B$ denotes that there is an automorphism mapping $A$ to $B$.

## 9. The Next Frontier

The New Extension Theorem 5.1 provides conditions which are sufficient to imply that $A \simeq_{\Delta_3^0} B$ and are virtually necessary, but we do not attempt to formally claim or prove it here. The following remarkable result suggests that something along these lines may be necessary and sufficient for *all* automorphisms of $\mathcal{E}$.

Define

$$\mathcal{S}(A) = \{W : (\exists V)[W \sqcup V = A\},$$

$$\mathcal{R}(A) = \{R : R \subseteq A \ \& \ R \text{ computable } \},$$

and $\mathcal{S}_{\mathcal{R}}(A)$ be the quotient structure of $\mathcal{S}(A)$ modulo $\mathcal{R}(A)$.

THEOREM 9.1 (Cholak-Harrington, ta). *If $A$ and $B$ are automorphic then*

$$\mathcal{S}_{\mathcal{R}}(A) \cong_{\Delta_3^0} \mathcal{S}_R(B).$$

The connection between Theorem 9.1 and the New Extension Theorem 5.1 and condition (T2) is this. For a fixed $\nu$ let $X_\nu$ ($Y_\nu$) be the set of elements which enter $A$ in state $\nu$ (not in state $\nu$). Now $X_\nu$ and $Y_\nu$ give a splitting of $A$. Theorem 9.1 says that even for the most general automorphism of $\mathcal{E}$ mapping $A$ to $B$ these sets must be in a $\Delta_3^0$-correspondence, just as the New Extension Theorem 5.1 says that a corresponding $\Delta_3^0$-condition is sufficient. These connections are far from being worked out, but they point the way toward the ultimate automorphism theorem.

## 10. Reflections on Computably Enumerable Sets

For sixty-five years we have intensively studied the Church-Turing Thesis and its characterization of computability. However, during the same period we have not studied a c.e. set foundationally as separate object, but only as the ranges of computable functions on $\omega$. The results for: (1) definable properties of $\mathcal{E}$; (2) automorphisms of $\mathcal{E}$: (3) c.e. sets applied to differential geometry, by Nabutovsky [1996a] and [1996b] and Nabutovsky and Weinberger [ta1] and [ta2]; suggest that a key factor in understanding c.e. sets is a dynamic approach which studies them from the point of view of properties like prompt simplicity or the $A >> B$ domination relation of Soare [Ta2] examining how fast elements enter one set $W_e$ in relation to elements entering other sets.

Virtually all our practical computing processes as well as many theoretical ones (like the search for local minima on manifolds) work as c.e. processes looking for an output, which cannot always be guaranteed ahead of time. It seems that more direct conceptual effort should be devoted to c.e. sets and particularly to their dynamic properties.

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