On the Turing Degrees of Minimal Index Sets

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Abstract

We study generalizations of shortest programs as they pertain to Schaefer's MIN* problem. We identify sets of m-minimal and T-minimal indices and characterize their truth-table and Turing degrees. In particular, we show $\mathrm{MIN}^{\mathrm{m}} \oplus \emptyset'' \equiv_{\mathrm{T}} \emptyset'''$, $\mathrm{MIN}^{\mathrm{T}^{(n)}} \oplus \emptyset^{(n+2)} \equiv_{\mathrm{T}} \emptyset^{(n+4)}$, and that there exists a Kolmogorov numbering ψ satisfying both $\mathrm{MIN}^{\mathrm{m}}_{\psi} \equiv_{\mathrm{tt}} \emptyset'''$ and $\mathrm{MIN}^{\mathrm{T}^{(n)}}_{\psi} \equiv_{\mathrm{T}} \emptyset^{(n+4)}$. This Kolmogorov numbering also achieves maximal truth-table degree for other sets of minimal indices. Finally, we show that the set of shortest descriptions, SD, is 2-c.e. but not co-2-c.e. Some open problems are left for the reader.

1 The MIN* problem

The set of shortest programs is

$$\text{f-MIN} := \{e : (\forall j < e)\,[\varphi_j \neq \varphi_e]\}.$$

In 1972, Meyer demonstrated that f-MIN admits a neat Turing characterization, namely f-MIN $\equiv_{\mathrm{T}} \emptyset''$ [10]. In Spring 1990 (according to the best recollection of the author), John Case issued a homework assignment with the following definition [1]:

$$\text{f-MIN}^* := \{e : (\forall j < e)\,[\varphi_j \neq^* \varphi_e]\},$$

where $=^*$ means equal except for a finite set. Case notes that f-MIN* is Σ_2-immune, although his assignment exclusively refers to the Σ_2-sets as "lim-r.e." sets. On October 1, 1996, six years after the initial homework assignment, Case introduced the set f-MIN* to Marcus Schaefer in an email.

The following year, Schaefer published a master's thesis on minimal indices [14], which became the first public account of f-MIN*. In his survey thesis, Schaefer proved that f-MIN$^* \oplus \emptyset' \equiv_{\mathrm{T}} \emptyset'''$, leaving open the tantalizing question of whether or not f-MIN $\equiv_{\mathrm{T}} \emptyset'''$. All that would be required to answer this question affirmatively is to show that f-MIN$^* \geq_{\mathrm{T}} \emptyset'$, care of Schaefer's result. This is the "MIN* problem." The reader is encouraged to attempt this reduction before proceeding. This concludes our historical remarks.

Our approach in this paper is to study c.e. sets in place of p.c. functions. This allows us to consider equivalence relations other than $=$ and $=^*$ which are especially natural for sets, namely:

Definition 1.1. For $n \geq 0$:

$$\begin{aligned}
\mathrm{MIN} &:= \{e : (\forall j < e)\, [W_j \neq W_e]\},\\
\mathrm{MIN}^* &:= \{e : (\forall j < e)\, [W_j \neq^* W_e]\},\\
\mathrm{MIN}^{\mathrm{m}} &:= \{e : (\forall j < e)\, [W_j \not\equiv_{\mathrm{m}} W_e]\},\\
\mathrm{MIN}^{\mathrm{T}^{(n)}} &:= \{e : (\forall j < e)\, [W_j \not\equiv_{\mathrm{T}^{(n)}} W_e]\}.
\end{aligned}$$

where $A \equiv_{\mathrm{T}^{(n)}} B$ is shorthand for $A^{(n)} \equiv_{\mathrm{T}} B^{(n)}$. If $n = 0$, we omit "(n)" from the notation. The sets above are called *minimal index sets.*

In Section 2, we generalize Schaefer's MIN* problem and obtain analogous results by characterizing the Turing degrees for the sets in Definition 1.1. We also pin down the complexity of the set of shortest descriptions, SD (see Definition 2.5). The primary lemma of Section 2, Lemma 2.10, turns out to be useful in both Theorem 2.16 and Lemma 3.3. A result on Σ_3-sets, Corollary 2.15, also follows from this lemma. In Section 3, we show that, in a formal sense, it will be difficult to prove the optimality of our results from Section 2. In particular, we show that there is a Kolmogorov numbering for which all of the sets in

Definition 1.1 simultaneously achieve maximum possible truth-table or Turing degree. Thus one must take into consideration Gödel numberings in order to prove any nontrivial upper bound on the degrees of MIN^*, $\mathrm{MIN}^{\mathrm{m}}$, or $\mathrm{MIN}^{\mathrm{T}^{(n)}}$.

Following notation in [16], we use $\equiv_{\mathrm{bT}}$ for bounded Turing equivalence, otherwise known as "weak truth-table" equivalence. When the Gödel numbering is relevant to a particular set, we shall include it as a subscript, as in MIN_φ. Notation not explained here can be found in [18]. For further background on minimal indices, we refer the reader to [21] and [14].

2 Turing characterizations

When squeezed gently, a fair amount of information can be extracted from minimal index sets. To show that $\emptyset^{(n)}$ reduces to a minimal index set, one first tries to achieve this (difficult) reduction with the aid of some oracle. By repeatedly substituting with successively weaker oracles, eventually one eliminates the oracle entirely (hopefully). Each time that a weaker oracle is introduced, a new reduction technique is required. We organize according to technique. Each section describes one or more reduction methods which pertain to oracles of particular strength.

2.1 Generic reductions

Lemma 2.2 shows how to "drop" a minimal index set "down one level." We demonstrate an especially short proof which is peculiar to $\mathrm{MIN}^{\mathrm{m}}$, however there is a canonical strategy which works for minimal index sets in general. The canonical strategy is presented in the proofs of (i) and (iv). (i) and (ii) first appeared in [14] and [10] for f-MIN and f-MIN^*, respectively. Although it is possible to prove Lemma 2.2 without the following theorem, we include it for illustrative purposes.

Theorem 2.1 ($\equiv_m$-Completeness Criterion, Jockusch et al. [6])**.** *Let $A \in \Sigma_3$ and $\emptyset'' \leq_{\mathrm{T}} A$. Then*

$$A \equiv_{\mathrm{T}} \emptyset''' \iff (\exists f \leq_{\mathrm{T}} A)\,(\forall e)\,\left[W_e \not\equiv_{\mathrm{m}} W_{f(e)}\right].$$

Lemma 2.2. *For $n \geq 0$,*

(I) $\mathrm{MIN} \oplus \emptyset' \geq_{\mathrm{T}} \emptyset''$,

(II) $\mathrm{MIN}^* \oplus \emptyset'' \geq_{\mathrm{T}} \emptyset'''$,

(III) $\mathrm{MIN}^{\mathrm{m}} \oplus \emptyset'' \geq_{\mathrm{T}} \emptyset'''$,

(IV) $\mathrm{MIN}^{\mathrm{T}^{(n)}} \oplus \emptyset^{(n+3)} \geq_{\mathrm{T}} \emptyset^{(n+4)}$.

Proof. (i). Let a be the minimal index for TOT, and let e be any index. Note that $W_e = W_x$ for exactly one x in

$$B := \{0, \ldots, e\} \cap \mathrm{MIN}.$$

Since

$$\{\langle j, e\rangle : W_j \neq W_e\} \in \Sigma_2,$$

we can enumerate all the indices $y \in B$ such that $W_y \neq W_e$ using a $\emptyset'$ oracle. Eventually, we enumerate all of the indices except for one. If the leftover index is a, then $W_e = W_a$, so $e \in \mathrm{TOT}$. Otherwise, $e \notin \mathrm{TOT}$. Thus, we can decide membership for a Π_2-complete set using only a $\mathrm{MIN} \oplus \emptyset'$ oracle. □

(ii). The argument in (iv) with COF substituted for HIGH^n yields the result, without taking into consideration other Gödel numberings (as was done in [14]). □

(iii). Define a $\overline{\mathrm{MIN}^{\mathrm{m}}}$-computable function f by

$$f(e) := (\mu i)\left[i \in \mathrm{MIN}^{\mathrm{m}} \quad \& \quad i > e\right].$$

Then

$$(\forall e)\left[W_e \not\equiv_{\mathrm{m}} W_{f(e)}\right].$$

Since $\overline{\mathrm{MIN}^{\mathrm{m}}} \in \Sigma_3$, it follows from the $\equiv_{\mathrm{m}}$-Completeness Criterion (Theorem 2.1) that

$$\mathrm{MIN}^{\mathrm{m}} \oplus \emptyset'' \equiv_{\mathrm{T}} \overline{\mathrm{MIN}^{\mathrm{m}}} \oplus \emptyset'' \equiv_{\mathrm{T}} \emptyset'''.$$ □

(iv). $\min^{\mathrm{T}^{(n)}}(e)$ denotes the function which computes the $\equiv_{\mathrm{T}^{(n)}}$-minimal index of e. We claim that

$$\min^{\mathrm{T}^{(n)}} \leq_{\mathrm{T}} \mathrm{MIN}^{\mathrm{T}^{(n)}} \oplus \emptyset^{(n+3)}.$$

Let a denote the $\mathrm{T}^{(n)}$-minimal index for $\emptyset'$. Since

$$\{\langle j, e\rangle : W_j \equiv_{\mathrm{T}^{(n)}} W_e\} \in \Sigma_{n+4},$$

we can enumerate the pairs of $\equiv_{\mathrm{T}^{(n)}}$-equivalent c.e. sets using a $\emptyset^{(n+3)}$ oracle.

For any index e, $W_e \equiv_{\mathrm{T}^{(n)}} W_x$ for exactly one x in

$$\{0, \ldots, e\} \cap \mathrm{MIN}^{\mathrm{T}^{(n)}}.$$

Since a unique x is guaranteed to exist, we have that $x = \min^{\mathrm{T}^{(n)}}(e)$ can be computed from a $\mathrm{MIN}^{\mathrm{T}^{(n)}} \oplus \emptyset^{(n+3)}$ oracle. This proves the claim.

Now since

$$\mathrm{HIGH}^n = \{e : W_e \equiv_{\mathrm{T}^{(n)}} \emptyset'\}$$

is Σ_{n+4}-complete [15],[18, Theorem XII.4.4], it suffices to determine, using a $\mathrm{MIN}^{\mathrm{T}^{(n)}} \oplus \emptyset^{(n+3)}$ oracle, whether a given index e is in HIGH^n. To do this, just compute $\min^{\mathrm{T}^{(n)}}(e)$, and check whether it is equal to a. □

□

The following arithmetic lower bounds are immediate consequences of Lemma 2.2, in light of the straightforward upper bounds from [21].

Corollary 2.3.

(I) $\mathrm{MIN} \in \Sigma_2 - \Pi_2$.

(II) $\mathrm{MIN}^* \in \Pi_3 - \Sigma_3$.

(III) $\mathrm{MIN}^{\mathrm{m}} \in \Pi_3 - \Sigma_3$.

(IV) $\mathrm{MIN}^{\mathrm{T}^{(n)}} \in \Pi_{n+4} - \Sigma_{n+4}$.

2.2 (Old)-timers

Prior to this work, the only technique which was successful in reducing a minimal index set by a second "level" was to use MIN queries to build a "timer" for the convergence of some function, thereby turning an enumerable object into something computable. Unlike the technique of Lemma 2.2, however, the "timer" method appears to be peculiar to the equivalence relation under consideration. We demonstrate this method in Lemma 2.4.

Meyer and Schaefer proved Lemma 2.4 for f-MIN and f-MIN* (respectively), but a similar proof works for both sets and functions.

Lemma 2.4 (Meyer [10], Schaefer [14])**.**

(I) $\text{MIN} \geq_{\text{bT}} \emptyset'$,

(II) $\text{MIN}^* \oplus \emptyset' \geq_{\text{T}} \emptyset''$.

Proof. (i). Let e be an index. We show how to decide whether $\varphi_e(e)\downarrow$ with a MIN oracle. Using the *s-m-n* Theorem, define a computable function f by

$$\varphi_{f(i)}(x) := \begin{cases} 1 & \text{if } \varphi_{i,x}(i)\downarrow, \\ \uparrow & \text{otherwise.} \end{cases} \tag{2.1}$$

Now $e \in K$ iff $W_{f(e)} \neq \emptyset$. $\varphi_{f(i)}(x)$ effectively counts the steps in computation $\varphi_i(x)$.

Let a be the minimal index of the function which diverges everywhere. Define a function $s : \text{MIN} - \{a\} \to \omega$ by

$$s(j) := \text{first } x \text{ found such that } \varphi_j(x)\downarrow,$$

and let

$$S(i) := \max_{\substack{j \leq i \\ (j \in \text{MIN} - \{a\})}} s(j).$$

Since $\varphi_{f(e)}$ agrees with some index in MIN $\cap \{0, \ldots, f(e)\}$, it must be the case that

$$\begin{aligned} W_{f(e)} \neq \emptyset &\iff W_{f(e)} \cap \{0, \ldots, S[f(e)]\} \neq \emptyset \\ &\iff \varphi_{e, S[f(e)]}(e)\downarrow . \end{aligned}$$

Since S is computable in MIN, we can decide $W_{f(e)} \neq \emptyset$. □

(ii). Recall that TOT $\equiv_{\mathrm{T}} \emptyset''$. Since $\overline{\mathrm{TOT}}$ is c.e. in $\emptyset'$, it suffices to enumerate TOT using a $\mathrm{MIN}^* \oplus \emptyset'$ oracle. Define computable functions f and g by

$$\varphi_{f(i)}(x) := \begin{cases} \langle x, (\mu s)\,(\forall y \leq x)\,[\varphi_{i,s}(y)\downarrow]\rangle & \text{if such an } s \text{ exists,} \\ \uparrow & \text{otherwise.} \end{cases}$$

$$\varphi_{g(i)}(x) := \begin{cases} \pi_2[\varphi_i(y)] & \text{for the first } y \geq x \text{ found such that } \varphi_i(y)\downarrow, \\ \uparrow & \text{if such a } y \text{ does not exist.} \end{cases}$$

where π_2 denotes projection in the second coordinate. Let a be the $=^*$-minimal index for the function which diverges everywhere. Define

$$A := \left\{ e : (\exists \langle j, N\rangle)\ \Big[j \in [\mathrm{MIN}^* - \{a\}] \cap \{0, \ldots, f(e)\} \quad \& \quad (\forall x)[\varphi_{e, \max\{N, \varphi_{g(j)}(x)\}}(x)\downarrow] \Big] \right\}. \tag{2.2}$$

We claim:

1. A is enumerable with a $\mathrm{MIN}^* \oplus \emptyset'$ oracle, and

2. $A = \mathrm{TOT}$.

Note that W_j is infinite when $j \in \mathrm{MIN}^* - \{a\}$, which makes $\varphi_{g(j)}$ a total function. The bracketed clause in (2.2) is therefore computable in $\mathrm{MIN}^* \oplus \emptyset'$, which proves (1).

If $e \in A$ then the universal clause in (2.2) is satisfied, so $e \in \mathrm{TOT}$. Conversely, assume $e \in \mathrm{TOT}$. Then $f(e) \in \mathrm{INF}$, so $f(e)$'s $=^*$-minimal index is not a. Let j be the $=^*$-minimal index for $f(e)$, choose n large enough so that

$$(\forall x > n)\left[W_j(x) = W_{f(e)}(x)\right],$$

and choose N large enough so that

$$(\forall x \leq n)\,[\varphi_{e,N}(x)\downarrow]\,.$$

Then for all x,

$$\max\{N, \varphi_{g(j)}(x)\} \geq \pi_2[\varphi_{f(e)}(x)],$$

because $\pi_2[\varphi_{f(e)}]$ is a nondecreasing function. Hence

$$(\forall x)\ [\varphi_{e,\max\{N,\varphi_{g(j)}(x)\}}(x)\downarrow],$$

so our selected pair $\langle j, N\rangle$ exhibits that $e \in A$. □

□

Definition 2.5 (Schaefer [14])**.**

$$\mathrm{SD} := \{e : (\forall j < e)\ [\varphi_j(0) \neq \varphi_e(0)]\}$$

is the "set of shortest descriptions."

We give one more example of the timer method. Unlike the other sets from Definition 1.1, SD does not sit properly inside a Σ_n or Π_n class, but rather in Δ_2. In particular, SD is 2-c.e.

Lemma 2.6 (Fortnow [4])**.** $\mathrm{SD} \notin \Sigma_1 \cup \Pi_1$.

Proof. $\mathrm{SD} \notin \Sigma_1$ follows immediately from the fact that SD is immune [14]. Suppose $\mathrm{SD} \in \Pi_1$. Let a be the smallest index such that $\varphi_a(0)\uparrow$. Define a computable function f by way of the *s*-*m*-*n* Theorem [18] and the following constant function:

$$\varphi_{f(x)}(y) := \begin{cases} (\mu t)\ [\varphi_{x,t}(0)\downarrow] & \text{if } \varphi_x(0)\downarrow, \\ \uparrow & \text{otherwise.} \end{cases}$$

Let

$$K_0 := \{e : \varphi_e(0)\downarrow\}.$$

K_0 is Σ_1-complete. Note that

$$\begin{aligned} e \in K_0 &\iff \varphi_{f(e)}(0)\downarrow \\ &\iff (\exists j \in [\{0,\ldots,f(e)\} \cap \mathrm{SD}] - \{a\}) \left[\varphi_j(0)\downarrow \quad \& \quad \varphi_{e,\varphi_j(0)}(0)\downarrow\right], \\ &\iff (\exists j \leq f(e)) \left[j \in \mathrm{SD} - \{a\} \quad \& \quad \varphi_j(0)\downarrow \quad \& \quad \varphi_{e,\varphi_j(0)}(0)\downarrow\right]. \end{aligned}$$

This means that $\overline{K_0} \in \Sigma_1$, since $j \in \mathrm{SD} - \{a\} \implies \varphi_j(0)\downarrow$. But that's a contradiction, because now K_0 is computable. □

Theorem 2.7 (Stephan [19])**.** SD *is 2-c.e. but not co-2-.c.e.*

Proof. First we show that SD is 2-c.e. via the following 2-c.e. algorithm $\{A_s\}_{s\in\omega}$. Let d be the least index such that $\varphi_d(0)\uparrow$, so that d is the unique member of SD with this property. We define $A_s(e)$ to make the right choice automatically for $e \leq d$. On input e with $e > d$, we start with $A_0(e) = 0$. If at some stage s, $\varphi_{e,s}(0)\downarrow = x$, then we take $A_s(e) = 1$. This will continue forever, unless at some stage $t > s$ we find that $\varphi_{j,t}(0)\downarrow = x$ for some $j < x$. In this case the algorithm would change its mind a second time: $A_t(e) := 0$. Thus whether or not $\varphi_e(0)\downarrow$, $\lim A_s(e) = \mathrm{SD}(e)$ and $\{A_s(e)\}$ changes its mind at most twice.

It remains to prove that $\overline{\mathrm{SD}}$ is not 2-c.e. Suppose that $\{B_s\}_{s\in\omega}$ approximates $\overline{\mathrm{SD}}$ while changing its mind at most twice. Let F denote the c.e. set

$$F := \{e : (\exists s,t)\,[s < t \quad \& \quad B_s(e) = 1 \quad \& \quad B_t(e) = 0 \quad \& \quad \varphi_e(0)\downarrow\}.$$

Now $F \subseteq \mathrm{SD}$. If F were infinite then we could find, using the Recursion Theorem, an index n satisfying

$$\varphi_n = \varphi_{(\mu e > n)\,[e \in F]}.$$

This would mean that $\varphi_e(0) = \varphi_n(0)$ for some $e > n$, contradicting the fact that $e \in \mathrm{SD}$.

Therefore F is finite. Let m be larger than both d and the greatest element of F. Now

$$\overline{\mathrm{SD}} - \{0, \ldots, m\} = \{e > m : (\exists s)\, [B_s(e) = 1]\}.$$

Indeed, if $\{B_s(e)\}$ were to change its mind a second time on some $e > m$, this would force $\varphi_e(0)\uparrow$ because $e \notin F$. But since $e > d$, we have $e \in \overline{\mathrm{SD}}$, and so $\{B_s\}$ has made a mistake in its approximation. This shows $\overline{\mathrm{SD}} \in \Sigma_1$, contrary to Lemma 2.6. It follows that $\overline{\mathrm{SD}}$ is not 2-c.e. $\square$

2.3 The Forcing Lowness Lemma

We show how to "drop" $\mathrm{MIN}^{\mathrm{T}^{(n)}}$ by a second "level." Lemma 2.10 is easiest to digest when we recall that LOW^0 is the set of indices with computable domains. The lemma gives slightly more than we need to prove the main theorem of this section, which is Theorem 2.16. The argument in Theorem 2.16 only depends on knowing the index $a_{\langle k,n\rangle}(0)$, however the entire countable sequence $a_{\langle k,n\rangle}(0), a_{\langle k,n\rangle}(1), \ldots$, as well as uniformity in n, will be required for Lemma 3.3.

We state a simple version of [13, Theorem 6.3] by Sacks for use in Lemma 2.10. Sacks does not explicitly mention uniformity in his original proof, however Soare does [18, Theorem VIII.3.1].

Theorem 2.8 (Sacks Jump Theorem [12])**.** *Let B be any set, and let S be c.e. in B' with $B' \leq_{\mathrm{T}} S$. Then there exists a B-c.e. set A with $A' \equiv_{\mathrm{T}} S$. Furthermore, an index for A can be found uniformly from an index for S.*

Definition 2.9.

$$\mathrm{LOW}^n := \{e : W_e \equiv_{\mathrm{T}^{(n)}} \emptyset\}.$$

Lemma 2.10 (forcing lowness)**.** *There exists a ternary computable function $a_{\langle k,n\rangle}(i)$ such that for every index k and any number i, $W_{a_{\langle k,n\rangle}(i)} \leq_{\mathrm{T}^{(n)}} W_k$. Furthermore:*

(I) $k \in \mathrm{LOW}^n \implies (\forall i)\, [a_{\langle k,n\rangle}(i) \in \mathrm{LOW}^n]$,

(II) $k \notin \mathrm{LOW}^n \implies (\forall i \neq j)\left[W_{a_{\langle k,n\rangle}(i)} \mid_{\mathrm{T}^{(n)}} W_{a_{\langle k,n\rangle}(j)}\right]$.

In either case, $a_{\langle k,n\rangle}(i) \in \mathrm{LOW}^{n+1}$ *for all* k*,* n*, and* i*.*

Proof. We shall combine finite injury ([18, Exercise VII.2.7], [5], [11]) with standard permitting ([18, Exercise VII.2.3], [2], [22]) by playing the Friedberg-Muchnik strategy under $(W_k)^{(n)}$. Our construction follows [16].

Given inputs n and k, we show how to effectively find $\emptyset^{(n)}$-c.e. sets $A_0, A_1, \ldots$ so that $A_0 = (W_{a_{\langle k,n\rangle}(0)})^{(n)}$, $A_1 = (W_{a_{\langle k,n\rangle}(1)})^{(n)}$, $\ldots$ etc. satisfy the conclusions of the theorem. If n is nonzero, then we can subsequently (and uniformly) find appropriate indices for c.e. sets by iteratively applying the Sacks Jump Theorem (Theorem 2.8). For clarity purposes, we adopt the following abbreviations:

$$B_i := \bigoplus_{j \neq i} A_j,$$

$$(B_i)_s := \bigoplus_{j \neq i} (A_j)_s,$$

where $(A_j)_0 \subseteq (A_j)_1 \subseteq \ldots$ is a $\emptyset^{(n)}$-enumeration for A_j.

If $k \in \mathrm{LOW}^n$, our construction will satisfy for all i,

$$Q_i : A_i \equiv_{\mathrm{T}^{(n)}} \emptyset,$$

and if $k \notin \mathrm{LOW}^n$, our construction will meet the requirements, for all i and e:

$$N_i : A_i \leq_{\mathrm{T}^{(n)}} W_k,$$

$$R_{\langle e,i\rangle} : A_i \neq \Psi_e^{B_i}.$$

In the following construction, we imagine Y to be the set $\emptyset^{(n)}$. We write Y in place of $\emptyset^{(n)}$ simply to emphasize that our algorithm is independent of the choice of oracle. Furthermore, our construction will be uniform in k. Let

$$C_k := (W_k)^{(n)} \oplus \omega.$$

Now C_k is c.e. in $\emptyset^{(n)}$, and an index for C_k (with $\emptyset^{(n)}$ oracle) can be found uniformly from

k. The "ω" is added into the definition of C_k just to ensure that the set is infinite. Since our construction will no longer refer to the value k, we abbreviate with $C := C_k$. Using the $\emptyset^{(n)}$-index for C, we can effectively find a 1:1 function $c \leq_{\mathrm{T}} \emptyset^{(n)}$ such that $c(0), c(1), c(2), \ldots$ is an enumeration of C.

Construction.

Stage $s = 0$. Define $r(\langle e, i\rangle, 0) = -1$ for all $\langle e, i\rangle$. Set $(A_i)_0 = \emptyset \oplus Y$ for all i.

Stage $s+1$ ($s+1$ is an i^{th} prime power). Choose the least $e \leq s$ such that

$$r(\langle e, i\rangle, s) = -1 \quad \& \quad (\exists \text{ even } x)\Big[x \in \omega^{[\langle e,i\rangle]} - (A_i)_s \quad \& \quad \Psi_{e,s}^{(B_i)_s}(x)\downarrow = 0 \\ \& \quad (\forall \langle z, j\rangle < \langle e, i\rangle)\,[r(\langle z, j\rangle, s) < x] \quad \& \quad c(s) \leq x\Big]. \quad (2.3)$$

If there is no such e, then do nothing and go to stage $s+2$. If e exists, then we say $R_{\langle e,i\rangle}$ *acts* at stage $s+1$, Perform the following steps.

Step 1. Enumerate x in A_i.

Step 2. Define $r(\langle e, i\rangle, s+1) = s+1$.

Step 3. For all $\langle z, j\rangle > \langle e, i\rangle$, define $r(\langle z, j\rangle, s+1) = -1$.

Step 4. For all $\langle z, j\rangle < \langle e, i\rangle$, define $r(\langle z, j\rangle, s+1) = r(\langle z, j\rangle, s)$.

When $r(\langle z, j\rangle, s+1)$ is reset to -1. we say that requirement $R_{\langle z,j\rangle}$ is *injured.*

Stage $s+1$ ($s+1$ is not a prime power). Do nothing. Get some coffee.

Claim 2.11. *For all i, $A_i \leq_{\mathrm{T}} C$.*

Proof. To decide whether $x \in A_i$, wait for a stage s such that all the elements of C below $x+1$ have been enumerated into C, i.e.,

$$C \upharpoonright x \subseteq \{c(0), c(1), \ldots, c(s)\}.$$

Such a stage s is guaranteed to exist, and the oracle C lets us identify when this occurs. The final clause of (2.3), "$c(s) \leq x$," ensures that no element $\leq x$ get enumerated into A_i

after stage s. Hence

$$x \in A_i \iff x \in (A_i)_{s+1}. \qquad \square$$

If $C \leq_{\mathrm{T}} \emptyset^{(n)}$, then by Claim 2.11, A_i is $\emptyset^{(n)}$-computable for every i. This proves case (i). It remains to consider case (ii).

Claim 2.12. *If requirement $R_{\langle e,i\rangle}$ acts at some stage $s+1$ and is never later injured, then requirement $R_{\langle e,i\rangle}$ is met and $r(\langle e,i\rangle, t) = s+1$ for all $t \geq s+1$.*

Proof. Suppose $R_{\langle e,i\rangle}$ acts at stage $s+1$ and say e is an i^{th} prime power. Then

$$\Psi_e^{(B_i)_s}(x)\downarrow = 0$$

for some $x \in (A_i)_{s+1}$. Since no $R_{\langle z,j\rangle}, \langle z,j\rangle < \langle e,i\rangle$ ever acts after stage $s+1$, it follows by induction on $t > s$ that $R_{\langle e,i\rangle}$ never acts again and $r(\langle e,i\rangle, t) = s+1$ for all $t > s$. Hence no $R_{\langle z,j\rangle}$, $\langle z,j\rangle > \langle e,i\rangle$, enumerates any $x \leq s$ into any A_j $(j \neq i)$ after stage $s+1$. Therefore,

$$B_i \upharpoonright s = (B_i)_s \upharpoonright s$$

and

$$\Psi_e^{B_i}(x)\downarrow = 0 \neq A_i(x). \qquad \square$$

Claim 2.13. *Assume $C >_{\mathrm{T}} \emptyset^{(n)}$. Then for every $\langle e,i\rangle$, requirement $R_{\langle e,i\rangle}$ is met, acts at most finitely often, and $r(\langle e,i\rangle) := \lim_s r(\langle e,i\rangle, s)$ exists.*

Proof. Fix $\langle e,i\rangle$ and assume the statement holds for all $R_{\langle z,j\rangle}$, $\langle z,j\rangle < \langle e,i\rangle$. Let v be the greatest stage when some such $R_{\langle z,j\rangle}$ acts, if ever, and $v = 0$ if none exists. Then $r(\langle e,i\rangle, v) = -1$, and this persists until some stage $s+1 > v$ (if ever) when $R_{\langle e,i\rangle}$ acts. If $R_{\langle e,i\rangle}$ acts at some stage $s+1$, then $R_{\langle e,i\rangle}$ becomes satisfied and never acts again. It then follows from Claim 2.12 that $r(\langle e,i\rangle, t) = s+1$ for all $t \geq s+1$.

Either way, $r(\langle e,i\rangle)$ exists and $R_{\langle e,i\rangle}$ acts at most finitely often. Now suppose that $R_{\langle e,i\rangle}$ is not met. Then

$$A_i = \Psi_e^{B_i}.$$

By stage v, at most finitely many elements $x \in \omega^{[\langle e,i\rangle]}$ have been enumerated in A_i. No further elements are enumerated from $\omega^{[\langle e,i\rangle]}$ because only requirement $R_{\langle e,i\rangle}$ can enumerate in this row. Let $x \in \omega^{[\langle e,i\rangle]} - (A_i)_v$ be such that $x > v$. Eventually there will be a stage s such that

$$\Psi_{e,s}^{(B_i)_s}(x)\downarrow = 0,$$

because $x \notin A_i$. Since x never becomes a witness that $R_{\langle e,i\rangle}$ is satisfied, it must be the permitting clause "$c(s) \leq x$" in (2.3) which prevents this from happening. Therefore

$$C \upharpoonright x = \{c(0), \ldots, c(s)\} \upharpoonright x.$$

Since x was chosen arbitrarily, we now have an algorithm to compute any finite initial segment of C. Our algorithm used only a $\emptyset^{(n)}$ oracle to compute the function c. Therefore $C \leq_{\mathrm{T}} \emptyset^{(n)}$, contrary to assumption. So requirement $R_{\langle e,i\rangle}$ must be met. □

Case (ii) is now satisfied because the requirements $R_{\langle e,i\rangle}$ are met. Finally,

Claim 2.14 (Soare [17])**.** *For every k, n, and i, we have $a_{\langle k,n\rangle}(i) \in \mathrm{LOW}^{n+1}$.*

Proof. We may assume $C >_{\mathrm{T}} \emptyset^{(n)}$ because otherwise the result follows immediately from Claim 2.11. Using the relativized *s*-*m*-*n* theorem, define a computable function f such that for all $Y \subseteq \omega$,

$$\Psi_{f(e)}^{Y}(x) := \begin{cases} 0 & \text{if } \Psi_e^Y(e)\downarrow, \\ \uparrow & \text{otherwise.} \end{cases}$$

$\Psi_{f(e)}^{Y}$ is either the constant zero function or diverges everywhere, depending on Y. Define a computable "witness" function w by

$$w(\langle e,i\rangle, s) := \begin{cases} \text{most recent member of } A_i \cap \omega^{[\langle e,i\rangle]} \text{ after stage } s, \text{ or} \\ \langle 0, \langle e,i\rangle\rangle \text{ if none exists.} \end{cases}$$

Since each requirement acts only finitely often (Claim 2.13), the limit

$$\hat{w}(e,i) := \lim_s w(\langle e,i\rangle, s)$$

exists and witnesses $\Psi_e^{B_i}[\hat{w}(e,i)] \neq A_i[\hat{w}(e,i)]$. Finally, define a sequence of functions $g_i \leq_{\mathrm{T}^{(n)}} \emptyset$ by

$$g_i(e,s) := \begin{cases} 1 & \text{if } \Psi_{f(e),s}^{(B_i)_s}\Big(w\left[\langle f(e),i\rangle, s\right]\Big)\downarrow = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We show that

$$\hat{g}_i(e) := \lim_s g_i(e,s) \tag{2.4}$$

is the characteristic function for $(B_i)'$, which implies that $(B_i)' \leq_{\mathrm{T}^{(n)}} \emptyset'$ by the Limit Lemma.

Let t be a large enough stage so that $R_{\langle f(e),i\rangle}$ never gets injured after stage t, and large enough so that $w(\langle f(e),i\rangle,\cdot)$ has settled, i.e.

$$(\forall s > t)\; (w[\langle f(e),i\rangle, s] = w[\langle f(e),i\rangle, t] = \hat{w}[f(e),i]).$$

For clarity, let $\tilde{w}$ denote the value $\hat{w}[f(e),i]$, and let v_s denote the function

$$v_s(x) := \Psi_{f(e),t}^{(B_i)_s}(x).$$

Now for all $s > t$, $g_i(e,s) = g_i(e,t)$, so the limit in (2.4) exists. Indeed, if $v_t(\tilde{w})\downarrow = 0$, and at some later stage s, $\neg\,[v_s(\tilde{w})\downarrow = 0]$, this would force our construction to find a new witness for $R_{\langle e,i\rangle}$, contradicting the fact that $\tilde{w}$ is the final witness. If, on the other hand, $\neg\,[v_t(\tilde{w})\downarrow = 0]$, then this computation on $\tilde{w}$ must be preserved forever, lest $\mathrm{R}_{\langle e,i\rangle}$ acts again.

Since $\hat{g}_i(e) = g_i(e,t)$, it follows that

$$\begin{aligned} \hat{g}_i(e) = 1 \quad &\iff \quad \Psi_{f(e),t}^{(B_i)_t}\Big(\hat{w}[f(e),i]\Big)\downarrow = 0 \\ &\iff \quad \Psi_{f(e)}^{B_i}\Big(\hat{w}[f(e),i]\Big)\downarrow = 0 \\ &\iff \quad \Psi_e^{B_i}(e)\downarrow. \end{aligned}$$

Therefore $\hat{g}_i$ is the characteristic function for $(B_i)'$. This proves $a_{\langle k,n\rangle}(j) \in \mathrm{LOW}^{n+1}$ for all $j \neq i$, as $a_{\langle k,n\rangle}(j)$ is the $\emptyset^{(n)}$-index for $A_j \leq_{\mathrm{T}} B_i$. Since i was chosen arbitrarily, we conclude that, in fact, $a_{\langle k,n\rangle}(i) \in \mathrm{LOW}^{n+1}$ for all $i \in \omega$. □

□

Let REC := LOW^0, the set of indices with computable domains. Since REC is Σ_3-complete [18, Corollary 3.6], we have the following corollary:

Corollary 2.15. *For every $A \in \Sigma_3$, there exists a computable function f such that*

$$\begin{aligned} x \in A &\implies f(x) \in \mathrm{REC}, \\ x \notin A &\implies f(x) \in \overline{\mathrm{REC}} \cap \mathrm{LOW}^1. \end{aligned}$$

We now apply Lemma 2.10 to minimal index sets. Our first application is the following:

Theorem 2.16. $\mathrm{MIN}^{\mathrm{T}^{(n)}} \oplus \emptyset^{(n+2)} \geq_{\mathrm{T}} \emptyset^{(n+3)}$.

Proof. Since LOW^n is Σ_{n+3}-complete, it suffices to determine membership in LOW^n using a $\mathrm{MIN}^{\mathrm{T}^{(n)}} \oplus \emptyset''$ oracle. On input k, first compute $a_{\langle k,n\rangle}(0)$, where $a_{\langle k,n\rangle}$ is the computable function defined in Lemma 2.10, and let c be the least index such that

$$W_c \equiv_{\mathrm{T}^{(n)}} \emptyset,$$

(i.e., $c \in \mathrm{LOW}^n$). We would like to know whether $\min^{\mathrm{T}^{(n)}}(k) = c$.

Let

$$e := a_{\langle k,n\rangle}(0),$$

and

$$S_e := \{0, \ldots, e\} \cap \mathrm{MIN}^{\mathrm{T}^{(n)}}.$$

There exists a unique $x \in S_e$ satisfying $W_x \equiv_{\mathrm{T}^{(n)}} W_e$, however unlike in Theorem 2.2(iv), we can not discover which one it is by direct enumeration because we are now missing the

$\emptyset^{(n+3)}$ oracle. So we use "double enumeration" instead. Since $e \in \mathrm{LOW}^{n+1}$, the set

$$Y_e := S_e \cap \{y : W_y \leq_{\mathrm{T}^{(n)}} W_e\}$$

is c.e. in $\mathrm{MIN}^{\mathrm{T}^{(n)}} \oplus \emptyset^{(n+2)}$ (since $A \leq_{\mathrm{T}^{(n)}} B$ is a $\Sigma_{n+2}^{A \oplus B'}$ relation). Let $Y_{e,t}$ denote the elements which have been added into Y_e after t steps of this enumeration. We remark that $Y_{e,t} \leq_{\mathrm{T}} \mathrm{MIN}^{\mathrm{T}^{(n)}} \oplus \emptyset^{(n+2)}$.

Claim 2.17. *Define a function Z from* $\mathrm{range}[a_{\langle\cdot,n\rangle}(0)]$ *to finite sets by*

$$Z(e) := Y_e \cap \{y : W_e \leq_{\mathrm{T}^{(n)}} W_y\}\,.$$

Then

(I) $Z \leq_{\mathrm{T}} \mathrm{MIN}^{\mathrm{T}^{(n)}} \oplus \emptyset^{(n+2)}$, *and*

(II) $Z(e) = \{\min^{\mathrm{T}^{(n)}}(e)\}$.

Proof. (ii) is immediate because $z \in Z(e)$ implies $W_z \equiv_{\mathrm{T}^{(n)}} W_e$, and $\min^{\mathrm{T}^{(n)}}(e)$ is the unique member of S_e with this property. It remains to compute $Z(e)$ with a $\mathrm{MIN}^{\mathrm{T}^{(n)}} \oplus \emptyset^{(n+2)}$ oracle. Note that when $y \in Y_{e,t}$, the relation

$$(\exists i \leq t)\,(\forall x)\left[\Psi_i^{(W_y)^{(n)}}(x)\!\downarrow \quad \& \quad \left(x \in (W_e)^{(n)} \iff \Psi_i^{(W_y)^{(n)}}(x) = 1\right)\right] \tag{2.5}$$

is in $\Pi_1^{\emptyset^{(n+1)}} = \Pi_{n+2}$ because $y \in \mathrm{LOW}^{n+1}$. Therefore knowing *a priori* that we are considering only members of $Y_{e,t}$, we can decide membership in (2.5) using the $\emptyset^{(n+2)}$ oracle.

The algorithm for Z is as follows. Assume that we have not yet converged by stage t. For each $y \in Y_{e,t}$, we check using $\emptyset^{(n+2)}$ whether y satisfies (2.5). If we find a $y \in Y_{e,t}$ satisfying (2.5), then we know $W_e \leq_{\mathrm{T}^{(n)}} W_y$, hence $Z(e) = \{y\}$, so the algorithm terminates. Otherwise we proceed similarly in stage $t+1$. Eventually we will discover a $y \in Y_e$ satisfying (2.5), namely $y = \min^{\mathrm{T}^{(n)}}(e)$.

We have glossed over one important detail of our algorithm, namely whether or not we can check for membership in (2.5) *uniformly in e*. In fact, we can. In order to make the algorithm uniform in e, we not only need to know that $(W_y)' \leq_{\mathrm{T}^{(n)}} \emptyset'$, but we also need to know explicitly what the reduction is so that we can make the correct queries to $\emptyset^{(n+2)}$ (regarding (2.5)).

Here are the missing details. When we enumerate y into Y_e, we automatically obtain a witness for $W_y \leq_{\mathrm{T}^{(n)}} W_e$, namely the index of this reduction. Using this witness, we can effectively find a second index witnessing $(W_y)' \leq_{\mathrm{T}^{(n)}} (W_e)'$. Finally, e is a special set of the form $a_{\langle \cdot, n\rangle}(0)$, and so Claim 2.14 gives a recipe for deciding membership in $(W_e)^{(n+1)}$ given $\emptyset^{(n+1)}$. □

By Lemma 2.10,

$$\begin{aligned} Z(e) = \{c\} \quad &\Longleftrightarrow \quad \min^{\mathrm{T}^{(n)}}(e) = c \\ &\Longleftrightarrow \quad a_{\langle k,n\rangle}(0) = e \in \mathrm{LOW}^n \\ &\Longleftrightarrow \quad k \in \mathrm{LOW}^n. \end{aligned}$$

Thus, membership in LOW^n is decidable in $\emptyset^{(n+2)} \oplus \mathrm{MIN}^{\mathrm{T}^{(n)}}$. □

2.4 Conclusion

We summarize the main results of this chapter in Corollary 2.18.

Corollary 2.18.

(I) $\mathrm{SD} \equiv_{\mathrm{bT}} \emptyset'$.

(II) $\mathrm{MIN} \equiv_{\mathrm{T}} \emptyset''$,

(III) $\mathrm{MIN}^* \oplus \emptyset' \equiv_{\mathrm{T}} \emptyset'''$.

(IV) $\mathrm{MIN}^{\mathrm{m}} \oplus \emptyset'' \equiv_{\mathrm{T}} \emptyset'''$.

(V) $\mathrm{MIN}^{\mathrm{T}^{(n)}} \oplus \emptyset^{(n+2)} \equiv_{\mathrm{T}} \emptyset^{(n+4)}$.

Proof. The upper bounds $\mathrm{MIN} \leq_{\mathrm{T}} \emptyset''$, $\mathrm{MIN}^* \oplus \emptyset' \leq_{\mathrm{T}} \emptyset'''$, etc. follow immediately from Corollary 2.3. It remains to show the lower bounds.

(i). Use the proof from Lemma 2.4(i), but in (2.1) make f check for convergence on 0 rather than i. □

(ii), (iii). Combine Lemma 2.2 with Lemma 2.4. □

(iv). Lemma 2.2. □

(v). Combine Lemma 2.2 with Theorem 2.16. □

□

It would be interesting to know whether or not the $\emptyset'$, $\emptyset''$, or $\emptyset^{(n+2)}$ oracle is necessary in any of the above reductions. Corollary 3.8 will show, in a formal sense, that a positive answer to this question will be difficult to prove.

3 A Kolmogorov numbering

For certain Gödel numberings, we can exactly determine the truth-table degree of MIN, MIN^*, and $\mathrm{MIN}^{\mathrm{m}}$ as well as the Turing degrees of $\mathrm{MIN}^{\mathrm{T}^{(n)}}$, and $\mathrm{MIN}^{\text{Thick-}*}$. The main result of this chapter, Theorem 3.8, provides a Kolmogorov numbering in which minimal index sets exactly characterize the Turing degrees $\mathbf{0}$, $\mathbf{0}'$, $\mathbf{0}''$, $\ldots$.

3.1 Numbering I

Lemma 3.2, restricted to f-MIN_ψ and f-MIN^*_ψ, was first proved by Schaefer [14]. He also mentions a Gödel ordering satisfying (i). The construction here is inspired by [14, Theorem 2.17]. For illustrative purposes, we consider the following operation on equivalence classes:

Definition 3.1. Let $\equiv_\alpha$ be an equivalence relation, and let $A, B \subseteq \omega$. Define the relation

$$A \equiv_{\text{Thick-}\alpha} B \iff (\forall n)\left[A^{[n]} \equiv_\alpha B^{[n]}\right],$$

and the corresponding set

$$\mathrm{MIN}^{\text{Thick-}\alpha} := \{e : (\forall j < e)\, [W_e \not\equiv_{\text{Thick-}\alpha} W_j]\}$$

Lemma 3.2. *There exists a Kolmogorov numbering ψ simultaneously satisfying:*

(i) $\mathrm{SD}_\psi \geq_{\mathrm{tt}} \emptyset'$,

(ii) $\mathrm{MIN}_\psi, \text{f-MIN}_\psi \geq_{\mathrm{tt}} \emptyset''$,

(iii) $\mathrm{MIN}^*_\psi, \text{f-MIN}^*_\psi \geq_{\mathrm{tt}} \emptyset'''$,

(iv) $\mathrm{MIN}^{\text{Thick-}*}_\psi \geq_{\mathrm{tt}} \emptyset'''$,

(v) $\mathrm{MIN}^{\text{Thick-m}}_\psi \geq_{\mathrm{tt}} \emptyset'''$, *and*

(vi) $\mathrm{MIN}^{\text{Thick-T}^{(n)}}_\psi \geq_{\mathrm{tt}} \emptyset^{(n+4)}$.

Proof. We first construct a Gödel numbering ψ satisfying (vi). We later argue that our construction can be modified to produce a Kolmogorov numbering satisfying all six parts of the lemma.

Let φ be any Gödel numbering, and let $n \geq 0$. We define the numbering ψ as follows. Define an increasing, computable function f by

$$f(0) := 0,$$
$$f(k+1) := 4[f(k)+1]+1,$$

Let $i \geq 0$. If $i = f(k)$ for some k, then we define $\psi_i := \varphi_k$. This makes ψ an effective ordering. Otherwise, for some k, $f(k) < i < f(k+1)$. In this case we define

$$\psi_i(\langle x, y\rangle) := \begin{cases} 1 & \text{if } [y - f(k) \text{ is odd} \quad \& \quad y = i \quad \& \quad \varphi_x(x)\downarrow], \\ 1 & \text{if } [y - f(k) \text{ is even} \quad \& \quad y = i-1 \quad \& \quad \varphi_k(x)\downarrow], \\ \uparrow & \text{otherwise.} \end{cases} \tag{3.1}$$

The functions $\psi_{f(k)+1}, \psi_{f(k)+3} \ldots, \psi_{4[f(k)+1]-1}$ code the halting set into distinct rows, and the remaining functions between $f(k)$ and $f(k+1)$ are used for comparisons.

It remains now only to show that

$$\mathrm{HIGH}^n_\varphi \leq_{\mathrm{tt}} \mathrm{MIN}^{\text{Thick-T}^{(n)}}_\psi,$$

because HIGH^n_φ is Σ_{n+4} complete [15],[18, Theorem XII.4.4]. Here we use the subscript "φ" to emphasize that we are considering HIGH^n with respect to the numbering φ.

We claim that

$$k \in \mathrm{HIGH}_{\varphi}^{n} \iff \left[\mathrm{MIN}_{\psi}^{\text{Thick-T}^{(n)}} \cap \{f(k)+2, f(k)+4, \ldots, 4f(k)+4\}\right] = \emptyset. \tag{3.2}$$

The claim follows by inspecting pairs of functions $\{\psi_i, \psi_{i+1}\}$. Indeed, assume $k \in \mathrm{HIGH}_{\varphi}^{n}$. Then for all rows y, including $y = f(k)+1$,

$$\left(\operatorname{dom} \psi_{f(k)+1}\right)^{[y]} \equiv_{\mathrm{T}^{(n)}} \left(\operatorname{dom} \psi_{f(k)+2}\right)^{[y]}.$$

Therefore

$$\operatorname{dom} \psi_{f(k)+1} \equiv_{\text{Thick-T}^{(n)}} \operatorname{dom} \psi_{f(k)+2},$$

which means that

$$f(k)+2 \notin \mathrm{MIN}_{\psi}^{\text{Thick-T}^{(n)}}.$$

Similarly,

$$f(k)+4, f(k)+6 \ldots, 4f(k)+4 \notin \mathrm{MIN}_{\psi}^{\text{Thick-T}^{(n)}},$$

which proves the first direction.

Conversely, assume that $k \notin \mathrm{HIGH}_{\varphi}^{n}$. Then for all $i \neq j$, with

$$i, j \in \{f(k)+1, f(k)+2, \ldots, 4f(k)+4\},$$

we have

$$\psi_i \not\equiv_{\text{Thick-T}^{(n)}} \psi_j.$$

This means that for $k \geq 1$,

$$[4f(k)+4] - f(k) = 3f(k)+4$$

distinct $\equiv_{\text{Thick-T}^{(n)}}$-equivalence classes are represented in

$$\{\psi_{f(k)+1}, \psi_{f(k)+2}, \ldots \psi_{4f(k)+4}\}. \tag{3.3}$$

It follows that at least

$$[3f(k)+4]-(f(k)+1)=2f(k)+3$$

of the indices from (3.3) are $\equiv_{\text{Thick-T}^{(n)}}$-minimal, since only those classes also represented in $\{\psi_0,\ldots,\psi_{f(k)}\}$ could be $\equiv_{\text{Thick-T}^{(n)}}$-nonminimal. Thus, any subset from

$$\{f(k)+1, f(k)+2,\ldots,4f(k)+4\}$$

with cardinality at least $f(k)+2$ must contain a $\equiv_{\text{Thick-T}^{(n)}}$-minimal index. In particular,

$$\left[\text{MIN}_\psi^{\text{Thick-T}^{(n)}} \cap \{f(k)+2, f(k)+4,\ldots,4f(k)+4\}\right] \neq \emptyset.$$

Hence we conclude that

$$\text{MIN}_\psi^{\text{Thick-T}^{(n)}} \geq_{\text{tt}} \emptyset^{(n+4)}.$$

We now describe separate orderings satisfying (i) – (v), and then we show that all six numberings can be combined together into a single Gödel numbering. Finally, we argue that this Gödel numbering can be made into an Kolmogorov numbering by ambiguously appealing to [14, Theorem 2.17].

The remaining, individual numberings are either identical or similar to the numbering ψ which we just constructed. For instance, the same ψ satisfies

$$\text{MIN}_\psi^{\text{Thick-m}} \geq_{\text{tt}} \emptyset'''.$$

In fact, we need only change HIGH_φ^n to

$$\text{mCOMP}_\varphi := \{e : W_e \equiv_{\text{m}} K\}$$

in the verification (3.2), and then the same proof works. For $\equiv_{\text{Thick-}*}$, $=^*$, and $=$, we use a different numbering, say ν, which is exactly like ψ except the condition "$\varphi_x(x)\downarrow$" is omitted from (3.1). To verify this numbering works, we swap either COF_φ or TOT_φ for HIGH_φ^n in

(3.2). For SD, we assume $\xi_0(0)\uparrow$ and substitute (3.1) with the constant functions

$$\xi_i(x) := \begin{cases} \langle i, 1\rangle & \text{if} \quad i \text{ is odd,} \\ \langle i-1, 1\rangle & \text{if} \quad [i \text{ is even} \quad \& \quad \varphi_k(k)\downarrow], \\ \uparrow & \text{otherwise.} \end{cases}$$

In the verification for SD, we replace HIGH^n_φ in (3.2) with the halting set complement, $\overline{K_\varphi}$.

We now merge the numberings ψ, ν, and ξ into a single Gödel numbering ρ satisfying (i) – (vi). All we do is change the p.c. functions filling the coding "gap" between $f(k)$ and $f(k+1)$, so that ψ fills the first gap, ν fills the second gap, ξ fills the third gap, ψ again fills the fourth, etc. Furthermore, we must repeat each φ_k function three times, so that each of numbering strategies may ask questions to it. For this reason, we let φ be a Kolmogorov numbering such that $\varphi_k = \varphi_{k+1} = \varphi_{k+2}$ whenever $k \equiv 0 \pmod 3$. We could settle for a Gödel numbering for the moment, but we'll need φ to be a Kolmogorov numbering anyway after the next paragraph.

We define

$$\rho_i := \varphi_k \quad \text{when } i = f(k) \text{ for some } k.$$

Otherwise, $f(k) < i < f(k+1)$ for some k. If $k \equiv 0 \pmod 3$ then we use the ψ strategy for i, if $k \equiv 1 \pmod 3$ we use the ν strategy for i, and if $k \equiv 2 \pmod 3$ we use the ξ strategy for k. So, for example, if $i = 3 \cdot 4567 + 1$, then

$$\rho_i(\langle x, y\rangle) := \begin{cases} 1 & \text{if} \quad [y - f(k) \text{ is odd} \quad \& \quad y = i]\,, \\ 1 & \text{if} \quad [y - f(k) \text{ is even} \quad \& \quad y = i-1 \quad \& \quad \varphi_k(x)\downarrow]\,, \\ \uparrow & \text{otherwise.} \end{cases}$$

We can now make truth-table queries to the appropriate minimal index sets, just as before.

Finally, we transform ρ into a Kolmogorov numbering. The idea is to enumerate a large number of φ_k's between each coding "gap" instead of just the one k from $f(k)$. In the s^{th} gap, we code a crib for φ_s in the same manner as we did with ρ. More formally we define, by induction,

$$g(0) := 0, \tag{3.4}$$

$$h(0) := 0, \tag{3.5}$$

$$g(k+1) := g(k) + h(k) + 2[g(k)+1], \tag{3.6}$$

$$h(k+1) := 2[h(k) + 2(g(k)+1)]. \tag{3.7}$$

Our new numbering is split into blocks $h(k) \le i < h(k+1)$ rather than $f(k) \le i < f(k+1)$ as before. For i with

$$h(k) \le i < h(k) + 2[g(k)+1],$$

we apply the familiar coding scheme from ρ (on φ_k), and for i with

$$h(k) + 2[g(k)+1] \le i < h(k+1),$$

we simply enumerate $\varphi_{g(k)}$ up to $\varphi_{g(k+1)-1}$. This construction is a Kolmogorov numbering by [14, Theorem 2.17], where this same induction appears. □

3.2 Numbering II

We build another Kolmogorov numbering, this time using Lemma 2.10.

Lemma 3.3. *There exists a Kolmogorov numbering ψ such that for all $n \ge 0$:*

(i) $\mathrm{MIN}^{\mathrm{m}}_{\psi} \ge_{\mathrm{tt}} \emptyset'''$.

(ii) $\mathrm{MIN}^{\mathrm{T}^{(n)}}_{\psi} \ge_{\mathrm{tt}} \emptyset^{(n+3)}$.

Proof. As in Lemma 3.2, we shall first construct a Gödel numbering ψ satisfying (i) and (ii), and we later argue that the construction can be modified so as to achieve a single Kolmogorov numbering.

Let φ be an arbitrary Gödel numbering, and assume $\langle \cdot, \cdot \rangle$ is a bijective pairing function satisfying $\langle 0, 0 \rangle = 0$. Let a be the computable function from Lemma 2.10, defined in terms of this ordering. Define a computable function f by

$$f(0) := 0,$$
$$f(k+1) := 2f(k) + 3.$$

The numbering ψ is defined as follows. Let C be an arbitrary computable set, and let ψ_0 be such that

$$\operatorname{dom}\psi_0 := C.$$

Let $i \geq 1$. If $i = f(\langle k, n\rangle)$ for some pair $\langle k, n\rangle$, then $\psi_i := \varphi_{\langle k,n\rangle}$. Otherwise, $f(\langle k, n\rangle) < i < f(\langle k, n\rangle + 1)$ for some $\langle k, n\rangle$. In this case,

$$\psi_i := \varphi_{a_{\langle k,n\rangle}(i)}.$$

Let LOW^n_φ and LOW^n_ψ denote the LOW^n indices in terms of φ-indices and ψ-indices, respectively.

We claim, for $\langle k, n\rangle > 0$,

$$\mathrm{MIN}^{\mathrm{T}^{(n)}}_\psi \cap \{f(\langle k, n\rangle) + 1, f(\langle k, n\rangle) + 2, \ldots, 2f(\langle k, n\rangle) + 2\} \neq \emptyset \quad \Longleftrightarrow \quad k \in \overline{\mathrm{LOW}^n_\varphi}.$$

Indeed, if $k \in \mathrm{LOW}^n_\varphi$, then $a_{\langle k,n\rangle}(i) \in \mathrm{LOW}^n_\varphi$ for all i, hence

$$\{f(\langle k, n\rangle) + 1, \ldots, 2f(\langle k, n\rangle) + 2\} \subseteq \mathrm{LOW}^n_\psi,$$

and so

$$\mathrm{MIN}^{\mathrm{T}^{(n)}}_\psi \cap \{f(\langle k, n\rangle) + 1, \ldots, 2f(\langle k, n\rangle) + 2\} = \emptyset.$$

Conversely, if $k \in \overline{\mathrm{LOW}^n_\varphi}$, then by definition of a, each of the ψ-indices

$$f(\langle k, n\rangle) + 1, \ldots, 2f(\langle k, n\rangle) + 2 \tag{3.8}$$

represents a distinct $\mathrm{T}^{(n)}$-degree. At most $f(\langle k, n\rangle) + 1$ degrees are represented with smaller indices, so at least one of the $f(\langle k,n\rangle)+2$ degrees in (3.8) must be minimal. That is,

$$\mathrm{MIN}_\psi^{\mathrm{T}^{(n)}} \cap \{f(\langle k,n\rangle)+1,\dots,2f(\langle k,n\rangle)+2\} \neq \emptyset.$$

Since LOW^n is Σ_{n+3}-complete, this proves that ψ satisfies (ii).

Similarly, for $k>0$,

$$\mathrm{MIN}_\psi^{\mathrm{m}} \cap \{f(\langle k,0\rangle)+1,\dots,2f(\langle k,0\rangle)+2\} \neq \emptyset \iff k \in \overline{\mathrm{LOW}_\varphi^0},$$

which shows that ψ satisfies (i). One can now transform φ into a Kolmogorov numbering by following the same procedure from Lemma 3.2, starting from (3.4). □

3.3 Truth-table apogee

We present a Kolmogorov numbering for which minimal index sets achieve maximal truth-table and Turing degrees.

Definition 3.4. Let K^ω be the c.e. set in which each row is the halting set; that is, for all k,

$$(K^\omega)^{[k]} := K.$$

Similarly, let $K^{(n)^\omega}$ be the c.e. set given by

$$\left(K^{(n)^\omega}\right)^{[i]} := K^{(n)}$$

for all i. Define

$$\text{Thick-COF} := \{e : W_e \equiv_{\text{Thick-}*} \omega\}$$
$$\text{Thick-mCOMP} := \{e : W_e \equiv_{\text{Thick-m}} K^\omega\}$$
$$\text{Thick-HIGH}^n := \left\{e : (W_e)^{(n)} \equiv_{\text{Thick-T}} K^{(n)^\omega}\right\}$$

Lemma 3.5. *Let $n \geq 0$. Then*

(I) Thick-COF *is* Π_4*-complete.*

(ii) Thick-mCOMP *is* Π_4*-complete.*

(iii) Thick-HIGH^n *is* Π_{n+5}*-complete.*

Proof. (i). Let $A \in \Pi_4$. Then there exists a relation $R \in \Sigma_3$ such that

$$x \in A \iff (\forall y)\, R(x, y).$$

Since COF is Σ_3-complete [18], there exists a computable function g such that $R(x, y)$ iff $W_{g(x,y)}$ is cofinite. Therefore

$$x \in A \iff (\forall y)\left[W_{g(x,y)} =^* \omega\right].$$

Define a computable function f by

$$\varphi_{f(x)}^{[y]} := \varphi_{g(x,y)}.$$

Then

$$\begin{aligned} W_{f(x)} \equiv_{\text{Thick-}*} \omega &\iff (\forall y)\left[W_{g(x,y)} =^* \omega\right] \\ &\iff x \in A, \end{aligned}$$

which makes Thick-COF Π_4-complete. □

(ii). Recall that mCOMP is Σ_3-complete [23], [18]. By an argument analogous to part (i), we have that Thick-mCOMP is Π_4-complete. □

(iii). We use the same reasoning a third time. Recall that

$$\text{HIGH}^n = \{e : W_e \equiv_{\text{T}^{(n)}} K\}$$

is Σ_{n+4}-complete [15][18]. By an argument analogous to part (i), we have that Thick-HIGH^n is Π_{n+5}-complete. □

□

Lemma 3.6. *Let $n \geq 0$.*

(I) $\mathrm{MIN}^{\text{Thick-}*} \oplus \emptyset''' \equiv_{\mathrm{T}} \emptyset''''$,

(II) $\mathrm{MIN}^{\text{Thick-m}} \oplus \emptyset''' \equiv_{\mathrm{T}} \emptyset''''$,

(III) $\mathrm{MIN}^{\text{Thick-T}^{(n)}} \oplus \emptyset^{(n+4)} \equiv_{\mathrm{T}} \emptyset^{(n+5)}$.

Proof. The same proof from Lemma 2.2(i) works here when we substitute the fact that either Thick-COF is Π_4-complete, Thick-mCOMP is Π_4-complete, or Thick-HIGH^n is Π_{n+5}-complete for the fact that TOT is Π_2-complete. □

Combining the orderings from Lemma 3.2 and Lemma 3.3 (using techniques from these lemmas), we obtain:

Theorem 3.7. *There exists a Kolmogorov numbering ψ satisfying*

(I) $\mathrm{SD}_\psi \geq_{\mathrm{tt}} \emptyset'$,

(II) $\mathrm{MIN}_\psi, \text{f-MIN}_\psi \geq_{\mathrm{tt}} \emptyset''$,

(III) $\mathrm{MIN}^*_\psi, \text{f-MIN}^*_\psi \geq_{\mathrm{tt}} \emptyset'''$,

(IV) $\mathrm{MIN}^{\mathrm{m}}_\psi \geq_{\mathrm{tt}} \emptyset'''$,

(V) $\mathrm{MIN}^{\mathrm{T}^{(n)}}_\psi \geq_{\mathrm{tt}} \emptyset^{(n+3)}$,

(VI) $\mathrm{MIN}^{\text{Thick-}*}_\psi \geq_{\mathrm{tt}} \emptyset'''$,

(VII) $\mathrm{MIN}^{\text{Thick-m}}_\psi \geq_{\mathrm{tt}} \emptyset'''$,

(VIII) $\mathrm{MIN}^{\text{Thick-T}^{(n)}}_\psi \geq_{\mathrm{tt}} \emptyset^{(n+4)}$.

Using the numbering from Theorem 3.7, together with Lemma 3.6 and Lemma 2.2, we can conclude the following.

Corollary 3.8. *There exists a Kolmogorov numbering ψ simultaneously satisfying:*

(I) $\mathrm{SD}_\psi \equiv_{\mathrm{tt}} \emptyset'$,

(II) $\mathrm{MIN}_\psi \equiv_{\mathrm{tt}} \text{f-MIN}_\psi \equiv_{\mathrm{tt}} \emptyset''$,

(iii) $\mathrm{MIN}^*_\psi \equiv_{\mathrm{tt}} \text{f-}\mathrm{MIN}^*_\psi \equiv_{\mathrm{tt}} \emptyset'''$,

(iv) $\mathrm{MIN}^{\mathrm{m}}_\psi \equiv_{\mathrm{tt}} \emptyset'''$,

(v) $\mathrm{MIN}^{\mathrm{T}^{(n)}}_\psi \equiv_{\mathrm{T}} \emptyset^{(n+4)}$,

(vi) $\mathrm{MIN}^{\text{Thick-*}}_\psi \equiv_{\mathrm{T}} \emptyset''''$,

(vii) $\mathrm{MIN}^{\text{Thick-m}}_\psi \equiv_{\mathrm{T}} \emptyset''''$, *and*

(viii) $\mathrm{MIN}^{\text{Thick-T}^{(n)}}_\psi \equiv_{\mathrm{T}} \emptyset^{(n+5)}$.

Some of the sets in Corollary 3.8 admit truth-table equivalence, while others have equivalence only for Turing degrees. It would be interesting to know whether or not the theorem holds when the Turing equivalences are replaced with truth-table equivalence. Note that tt-equivalence is the best we can do because none of these minimal index sets btt-reduce to the halting set [3, Corollary 4.7].

4 Open questions

4.1 Is $\mathrm{MIN}^{\mathrm{T}} \equiv_{\mathrm{T}} \emptyset''''$?

We conjecture that Corollary 3.8 does not hold for arbitrary Gödel numberings. In particular, we conjecture that Corollary 2.18 is optimal in the following sense:

Conjecture 4.1. *Let* $n \geq 0$.

(i) *There exists a Gödel numbering* φ *such that* $\mathrm{MIN}^*_\varphi \not\geq_{\mathrm{T}} \emptyset'$.

(ii) *There exists a Gödel numbering* φ *such that* $\mathrm{MIN}^{\mathrm{m}}_\varphi \oplus \emptyset' \not\geq_{\mathrm{T}} \emptyset''$.

(iii) *There exists a Gödel numbering* φ *such that* $\mathrm{MIN}^{\mathrm{T}^{(n)}}_\varphi \oplus \emptyset^{(n+1)} \not\geq_{\mathrm{T}} \emptyset^{(n+2)}$.

Even showing $\mathrm{MIN}^{\mathrm{m}}_\varphi \not\geq_{\mathrm{T}} \emptyset''$ or $\mathrm{MIN}^{\mathrm{T}}_\psi \not\geq_{\mathrm{T}} \emptyset''$ for some Gödel numberings φ, ψ would prove that $\mathrm{MIN}^{\mathrm{m}}$ and $\mathrm{MIN}^{\mathrm{T}}$ do not have fixed Turing degrees.

All of the initial information in a $=^*$ set can be faulty [14], so intuitively one needs a halting set oracle to extract useful information from MIN^*. Similarly $\mathrm{MIN}^{\mathrm{m}}$ and $\mathrm{MIN}^{\mathrm{T}}$ presume knowledge of total functions, which suggests that $\emptyset'' \equiv_{\mathrm{T}} \mathrm{TOT}$ is undecidable relative to each of these sets. The difficulty in constructing the necessary numberings for Conjecture 4.1 is revealed by considering a simpler problem where we try to find *any* $A \in \Sigma_3$ satisfying:

$$A \oplus \emptyset' \equiv_{\mathrm{T}} \emptyset''',$$
$$A \not\geq_{\mathrm{T}} \emptyset'.$$

The existence of such an A follows from a deep result by Lerman [9, Theorem 1.2]. Making this construction work with $A = \mathrm{MIN}^*_\varphi$ for some *Gödel* numbering φ can only be more complicated.

If Conjecture 4.1 holds, then minimal index sets are (possibly the first) natural examples of sets which are not Turing equivalent to any of the canonical Σ_n-complete sets. If Conjecture 4.1 fails, then minimal index sets are a new and remarkable characterizations of the Turing degrees $\mathbf{0}'$, $\mathbf{0}''$, $\mathbf{0}'''$, ...

One approach to solving the MIN* problem is to look first at the related problem of $\mathrm{MIN}^{\mathrm{m}}$. This approach is promising because it has not received much attention. It is also promising for mathematical reasons. We now sing praises of $\mathrm{MIN}^{\mathrm{m}}$. If indeed $\mathrm{MIN}^{\mathrm{m}} \oplus \emptyset'' \equiv_{\mathrm{T}} \emptyset'''$ and $\mathrm{MIN}^* \oplus \emptyset' \equiv_{\mathrm{T}} \emptyset'''$ are both optimal results (in the sense of Conjecture 4.1), then it seems easy to find a numbering φ in which $\mathrm{MIN}^{\mathrm{m}}_\varphi$ avoids (merely) the cone of degrees above $\emptyset''$, when compared to the (daunting) task of forcing MIN^*_φ to avoid the cone above $\emptyset'$. The second reason to take up $\mathrm{MIN}^{\mathrm{m}}$ is for the elegance and brevity of results which are unique to $\mathrm{MIN}^{\mathrm{m}}$. The $\equiv_{\mathrm{m}}$-Fixed Point Theorem [6],

$$f \leq_{\mathrm{T}} \emptyset'' \implies (\exists e)\ [W_e \equiv_{\mathrm{m}} W_{f(e)}],$$

immediately gives optimal immunity for $\mathrm{MIN}^{\mathrm{m}}$ (namely $\mathrm{MIN}^{\mathrm{m}}$ is Σ_3-immune [21, Theorem 3.1.3]). This means that immunity for $\mathrm{MIN}^{\mathrm{m}}$ does not depend on the choice of Gödel numbering, which is not true for MIN, MIN*, or $\mathrm{MIN}^{\mathrm{T}^{(n)}}$ [20]. Furthermore, in contrast to other minimal index sets, our purported optimal result for the Turing degree of $\mathrm{MIN}^{\mathrm{m}}$, Lemma 2.2(iii), follows directly from an $\equiv_{\mathrm{m}}$-Completeness Criterion (Theorem 2.1). Finally, we have a satisfying proof of the fact that $\mathrm{MIN}^{\mathrm{m}}_{\psi} \equiv_{\mathrm{tt}} \emptyset'''$ for some Kolmogorov numbering ψ (Theorem 3.8). This same argument finds only a Turing degree for $\mathrm{MIN}^{\mathrm{T}^{(n)}}_{\psi}$.

4.2 Truth table degrees

Meyer's original question from 1972 remains open: is f-MIN $\equiv_{\mathrm{tt}} \emptyset''$ [10]? A reduction f-MIN $\geq_{\mathrm{bT}} \emptyset''$ would suffice to show f-MIN $\equiv_{\mathrm{tt}} \emptyset''$, if it were the case that $\emptyset' \leq_{\mathrm{tt}}$ MIN [14, Section 8]. Similarly, Schaefer asks, is SD $\equiv_{\mathrm{tt}} \emptyset'$ [14]? The fact that the Kolmogorov strings are tt-complete for any Kolmogorov numbering φ [8] but that we don't know this to be true for its cousin SD indicates that there is still a bit to learn about the similarities between randomness and shortest descriptions.

4.3 MIN vs. f-MIN

Recall that MIN $\equiv_{\mathrm{T}} \emptyset'' \equiv_{\mathrm{T}}$ f-MIN (Corollary 2.18 and [10]). What can be said about stronger reductions? We know that there exists a Kolmogorov numbering ψ such that $\mathrm{MIN}_{\psi} \equiv_{\mathrm{tt}} \text{f-MIN}_{\psi}$ (Theorem 3.8(ii)), and for any numbering φ, there is a Gödel numbering ψ such that $\text{f-MIN}_{\varphi} \not\equiv_{\mathrm{btt}} \text{f-MIN}_{\psi}$ [7]. Hence there is a Gödel numbering ν such that $\mathrm{MIN}_{\varphi} \not\equiv_{\mathrm{btt}} \text{f-MIN}_{\nu}$. But do there exists any Gödel numberings φ and ψ such that $\mathrm{MIN}_{\varphi} \equiv_{\mathrm{btt}} \text{f-MIN}_{\psi}$? Given a Gödel numbering φ, does there always exist a Gödel numbering ψ such that $\mathrm{MIN}_{\varphi} \equiv_{\mathrm{tt}} \text{f-MIN}_{\psi}$?

References

[1] John Case. personal communication.

[2] J. C. E. Dekker. A theorem on hypersimple sets. *Proc. Amer. Math. Soc.*, 5:791–796, 1954. ISSN 0002-9939.

[3] Stephen Fenner and Marcus Schaefer. Bounded immunity and btt-reductions. *MLQ Math. Log. Q.*, 45(1):3–21, 1999. ISSN 0942-5616.

[4] Lance Fortnow. personal communication.

[5] Richard M. Friedberg. Two recursively enumerable sets of incomparable degrees of unsolvability (solution of Post's problem, 1944). *Proc. Nat. Acad. Sci. U.S.A.*, 43: 236–238, 1957.

[6] C. G. Jockusch, Jr., M. Lerman, R. I. Soare, and R. M. Solovay. Recursively enumerable sets modulo iterated jumps and extensions of Arslanov's completeness criterion. *J. Symbolic Logic*, 54(4):1288–1323, 1989. ISSN 0022-4812.

[7] Jefim Kinber. On btt-degrees of sets of minimal numbers in Gödel numberings. *Z. Math. Logik Grundlagen Math.*, 23(3):201–212, 1977.

[8] Martin Kummer. On the complexity of random strings (extended abstract). In *STACS 96 (Grenoble, 1996)*, volume 1046 of *Lecture Notes in Comput. Sci.*, pages 25–36. Springer, Berlin, 1996.

[9] Manuel Lerman. The existential theory of the uppersemilattice of turing degrees with least element and jump is decidable. Draft at `http://www.math.uconn.edu/~lerman/eth-jump.pdf`.

[10] Albert R. Meyer. Program size in restricted programming languages. *Information and Control*, 21:382–394, 1972. ISSN 0890-5401.

[11] A. A. Mučnik. On the unsolvability of the problem of reducibility in the theory of algorithms. *Dokl. Akad. Nauk SSSR (N.S.)*, 108:194–197, 1956. ISSN 0002-3264.

[12] Gerald E. Sacks. Recursive enumerability and the jump operator. *Trans. Amer. Math. Soc.*, 108:223–239, 1963. ISSN 0002-9947.

[13] Gerald E. Sacks. *Degrees of unsolvability.* Princeton University Press, Princeton, N.J., 1963.

[14] Marcus Schaefer. A guided tour of minimal indices and shortest descriptions. *Arch. Math. Logic*, 37(8):521–548, 1998. ISSN 0933-5846.

[15] Steven Schwarz. *Quotient lattices, index sets, and recursive linear orderings.* PhD thesis, University of Chicago, 1982.

[16] Robert I. Soare. *Computability Theory and Applications*. Second edition. Revised edition of [18], currently unpublished.

[17] Robert I. Soare. The Friedberg-Muchnik theorem re-examined. *Canad. J. Math.*, 24: 1070–1078, 1972. ISSN 0008-414X.

[18] Robert I. Soare. *Recursively enumerable sets and degrees*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1987. ISBN 3-540-15299-7. A study of computable functions and computably generated sets.

[19] Frank Stephan. personal communication.

[20] Jason Teutsch and Frank Stephan. Immunity and hyperimmunity for generalized random strings. submitted.

[21] Jason R. Teutsch. *Noncomputable Spectral Sets*. PhD thesis, Indiana University, 2007.

[22] C. E. M. Yates. Three theorems on the degrees of recursively enumerable sets. *Duke Math. J.*, 32:461–468, 1965. ISSN 0012-7094.

[23] C. E. M. Yates. On the degrees of index sets. *Trans. Amer. Math. Soc.*, 121:309–328, 1966. ISSN 0002-9947.