A savings paradox for integer-valued gambling strategies

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Abstract

Under the assumption that wagers remain integer-valued, as would happen in most casinos, we identify the following bizarre situation: there exists a sequence of coin flips X such that some effective gambler manages to accumulate arbitrary wealth by betting on X, however any such gambler goes bankrupt whenever he tries to take his winnings outside the casino.

1 The savings trick

What good is money if you can't spend it?

As one might expect, a successful real-valued gambler can, in the long run, spend any sum of capital without damaging her gambling prospects inside a casino. On the other hand, if we restrict wagers to integers, then "no bets are off." Indeed, there exists a situation in which an integer-valued gambling strategy can gamble its way to arbitrary wealth inside a casino but goes bankrupt whenever it tries to spend its accumulated winnings (see Section 2). We review the Savings Trick for real-valued gambling strategies and then explain why this trick doesn't work for integer-valued gambling strategies. References [1] and [2] provide background on restricted betting strategies.

Notation. We shall focus on gambling strategies for betting on coin flips, and throughout this discussion we always assume that strategies are computable functions. More formally, a *gambling strategy* M is a computable function from finite strings of coin flips to nonnegative reals satisfying the *fairness condition*:

$$M(\sigma) = \frac{M(\sigma\mathtt{h}) + M(\sigma\mathtt{t})}{2}$$

where $M(\sigma)$ gives the gambler's *capital* after a finite series of coin flips $\sigma \in \{\mathtt{h}, \mathtt{t}\}^*$, "h" stands for heads, and "t" stands for tails. $M(\sigma\mathtt{h}) - M(\sigma)$ is *M's wager at σ*. If M's wager at σ is positive, we say that *M bets on heads at σ* and if M's wager at σ is negative, we say that *M bets on tails at σ*. A gambling strategy is called *integer-valued* if its wagers are always integers, and *real-valued* if its wagers can be any real number [1]. A gambling strategy *succeeds* on a binary sequence X if $\limsup_n M(X \restriction n) = \infty$. Here $X \restriction n$ denotes the length n prefix of X, and $X(n)$ will denote the n^{th} coin flip in X. ϵ denotes the empty string.

Over time, a gambling strategy may move capital into a permanent savings account. The gambling strategy cannot apply this money to future wagers, and one can imagine that the gambler reserves these funds for external purchases. In the definition below, the "savings function" describes the gambler's savings account balance.

Definition. A computable function f mapping $\{\mathtt{h}, \mathtt{t}\}^*$ to reals is called a *savings function* if for all binary strings σ and τ, σ is a prefix of τ implies $f(\sigma) \leq f(\tau)$. A *gambling-and-savings strategy* consists of a gambling strategy and a savings function. A gambling-and-savings strategy $\langle S, f\rangle$ *succeeds* on a binary sequence X if:

(I) $\lim_n f(X \restriction n) = \infty$, and

(II) $|S$'s wager at $\sigma| \leq S(\sigma) - f(\sigma)$ for all prefixes σ of X.

Let $\langle S, f\rangle$ be a gambling-and-savings strategy. We call $S(\sigma) - f(\sigma)$ the *wagerable capital* of $\langle S, f\rangle$ at σ. Following the terminology for gambling strategies, we say that $\langle S, f\rangle$ is *integer-valued* if S is integer-valued, *real-valued* if S is real-valued, and we say that an integer-valued savings-and-gambling strategy is *bankrupt* whenever it has less than \$1 in wagerable capital. An integer-valued gambling-and-savings strategy which is bankrupt at position σ cannot succeed on any sequence extending σ.

The following result is folklore, see [3].

Savings Trick. *If some real-valued gambling strategy succeeds on a sequence X, then some real-valued gambling-and-savings strategy also succeeds on X.*

Proof. Let M be a real-valued gambling strategy, and suppose that M succeeds on X. We will construct a gambling-and-savings strategy $\langle S, f\rangle$ which always bets the same fraction of its (wagerable) capital as M but occasionally puts money into savings. Each time S accumulates more than a dollar of capital, S puts a dollar into savings and implements a scaled copy of M's strategy on its remaining wagerable capital. Since M's capital goes to infinity over the sequence X, $\langle S, f\rangle$ will have infinitely many opportunities to sock away a dollar and hence will also succeed.

In more detail, initially $f(\epsilon) = 0$ and $S(\epsilon) = M(\epsilon)$. At each position σ,

$$S\text{'s wager at } \sigma = \frac{M\text{'s wager at } \sigma}{M(\sigma)} \cdot [S(\sigma) - f(\sigma)],$$

and for $a \in \{\mathtt{h}, \mathtt{t}\}$,

$$f(\sigma a) = \begin{cases} f(\sigma) & \text{if } S(\sigma a) - f(\sigma a) \leq 1, \text{ and} \\ f(\sigma) + 1 & \text{if } S(\sigma a) - f(\sigma a) > 1. \end{cases}$$

At the positions where $\langle S, f\rangle$ does not save, $S - f$ is positive and increases by the same proportion as M. Since $\limsup M(X \upharpoonright n) = \infty$, $[S - f](X \upharpoonright n)$ eventually becomes large enough that savings increase by \$1. Since this increase happens infinitely many times, $\langle S, f\rangle$ also succeeds. □

Combining the Savings Trick with the fact that a gambling-and-savings strategy's (total) capital is always bounded below by its savings, we see that the set of sequences on which some real-valued gambling strategy succeeds does not depend on whether we use "lim" or "lim sup" in the definition of succeeds. The same is true for integer-valued gambling strategies, albeit for a different reason. This fact was noted in a footnote of [2], however we include a short proof here to make this discussion self-contained.

Proposition. *The class of sequences where some integer-valued gambling strategy succeeds does not change if we replace "lim sup" with "lim" in the definition of succeeds.*

Proof. Suppose that some integer-valued gambling strategy M succeeds lim sup but not lim on a sequence X. Then M must return to some integer capital value infinitely often, lest $\liminf M(X \upharpoonright n) = \infty$. Let \$$k$ be the least amount of capital which M possesses infinitely often. Define a further integer-valued gambling strategy A which bets zero dollars except when M reaches capital \$$k$, at which point A bets a dollar on the same outcome as M. Since M's capital only drops below \$$k$ finitely often, this is almost always a sure bet for A, and since M visits capital \$$k$ infinitely often, A succeeds lim provided that her initial capital is sufficient. $\square$

Unlike the Savings Trick for real-valued gambling strategies, we do not construct the successful lim strategy from the Proposition uniformly in the lim sup one because we do not know *a priori* which value of k is correct or even whether the lim sup strategy will infinitely often visit some value k at all. Moreover, as we show in the next section, the lim successful gambler's capital may grow slower than any computable function.

2 A savings paradox

We show that the Savings Trick does not work when wagers are restricted to integers.

Theorem. *There exists a sequence on which some integer-valued gambling strategy succeeds but no integer-valued gambling-and-savings strategy does.*

Proof. We shall show that there is gambling strategy M which succeeds on a sequence X and always bets \$1 on heads while no integer-valued gambling-and-savings strategy succeeds on X. Assume that M's initial capital is at least \$1, and let $\langle S_0, f_0\rangle, \langle S_1, f_1\rangle, \langle S_2, f_2\rangle, \dots$ be a (noneffective) list of all integer-valued gambling-and-savings strategies. The basic module for bankrupting a single gambling-and-savings strategy $\langle S_e, f_e\rangle$ is the following. We may assume that S_e never bets on tails because if he does, then X can hurt S_e while at the same time helping M. Roughly speaking, our construction helps M except at positions where $\langle S_e, f_e\rangle$ bets a sufficiently greater fraction of his (wagerable) capital than M does. Our overall goal is to reduce the ratio $(S_e - f_e)/M$, and we guarantee that this ratio eventually drops below 1 if $\langle S_e, f_e\rangle$ saves enough money. Once M has more capital available to wager than $\langle S_e, f_e\rangle$, X can bankrupt $\langle S_e, f_e\rangle$ without bankrupting M simply by hurting S_e each time he makes a nonzero wager.

Without loss of generality, we may assume that $S_e(\epsilon) \geq f_e(\epsilon)$ since gambling-and-savings strategies which do not satisfy this initial condition are already bankrupt. Moreover any gambling-and-savings strategy which goes bankrupt thereafter cannot succeed, hence our finite extension construction focuses exclusively on the cases where $S_e(\sigma) - f_e(\sigma)$ is nonnegative. Similarly, we ignore any gambling-and-savings strategy once it places a wager exceeding its wagerable capital. Throughout this proof we implicitly round the functions $S_e - f_e$ and M down to the nearest integer because only the integer part of these quantities can ever be used for gambling.

In order to construct the desired sequence X, we will make use of three auxiliary functions. Let w be the computable "high water mark" mapping finite series of coin flips to nonnegative integers via

$$w(\sigma) = \max\{M(\tau)\colon \tau \text{ is a prefix of } \sigma\}.$$

The next two integer-indexed functions, q and r, map finite series of coin flips to positive integers by way of a modified Division Algorithm. $q_e(\sigma)$ and $r_e(\sigma)$ are defined to be the unique positive integers, when they exist, such that

$$S_e(\sigma) - f_e(\sigma) = q_e(\sigma) \cdot M(\sigma) - r_e(\sigma) \quad \text{and} \quad 0 < r_e(\sigma) \leq M(\sigma). \tag{1}$$

These integers are guaranteed to exist whenever M's capital is nonzero, and we shall argue that this condition is met on every prefix of the sequence X.

We define X by finite extensions. Assume that the length n prefix of X, which we call σ throughout the remainder of this proof, has already been defined. We define the length $n+1$ prefix of X, which we call σ', as follows. Let e be the least index less than or equal to $w(\sigma)$, if it exists, such that S_e's wager at σ is not $q_e(\sigma) - 1$. We say that S_e *receives attention at position* n. Define

$$X(n+1) = \begin{cases} \mathtt{t} & \text{if } S_e\text{'s wager at } \sigma \text{ exceeds } q_e(\sigma) - 1, \\ \mathtt{h} & \text{if } S_e\text{'s wager at } \sigma \text{ is less than } q_e(\sigma) - 1. \end{cases}$$

If no such e exists, then $X(n+1) = \mathtt{h}$. By convention S_e's wager at σ is positive iff S_e bets on heads at σ, and therefore X hurts S_e whenever S_e both bets on tails and receives attention.

We first verify that q_e and r_e are defined on every prefix of X. For simplicity of expression, we represent initial quantities in this paragraph with the following symbols:

$$q = q_e(\epsilon), \qquad r = r_e(\epsilon), \qquad m = M(\epsilon), \qquad s = S_e(\epsilon) - f_e(\epsilon).$$

Since $m \geq 1$ and $s \geq 0$, we can let q be the least positive integer such that $s < qm$, and we let $r = qm - s$. Then $0 < r \leq m$ as in (1). Now assume that q_e and r_e are defined on σ and $M(\sigma) \geq 1$. If no index receives attention at position n, then $X(n+1) = \mathtt{h}$, so $M(\sigma') \geq 2$, and therefore the argument from the base case shows that q_e and r_e are defined on σ'. On the other hand, suppose that e receives attention at n. Note that if $M(\sigma) = 1$, then $\langle S_e, f_e\rangle$ has at most $q_e(\sigma) - 1$ capital to wager at σ, and therefore $X(n+1) = \mathtt{h}$. Consequently, $M(\sigma') \geq 2$. Otherwise $M(\sigma) \geq 2$, and in this case M also cannot go broke because M wagers only one dollar at each position. Either way, we obtain values for $q(\sigma')$ and $r(\sigma')$ just as in the base case.

In the next six paragraph, we employ the following abbreviations:

$$\begin{aligned} q &= q_e(\sigma), & r &= r_e(\sigma), & m &= M(\sigma), & s &= S_e(\sigma) - f_e(\sigma), \\ q' &= q_e(\sigma'), & r' &= r_e(\sigma'), & m' &= M(\sigma'), & s' &= S_e(\sigma') - f_e(\sigma'), \end{aligned}$$

We show that whenever e receives attention at position n, one of the two Things below happens.

(I) $q' = q$ and $m' - r' < m - r$, or

(II) $q' < q$.

Suppose that S_e's wager at σ is greater than $q-1$, say $(q-1)+k$, for some $k \geq 1$. By definition of X, S_e and M lose capital on this bet, so $m' = m - 1$ and

$$s' = s - [(q-1)+k] = (qm - r) - q - k + 1 = q(m-1) - (r + k - 1).$$

If $r + k \leq m$ we have $q' = q$ and $m' - r' = (m - r) - k$, satisfying Thing (I). If not, then $s' = q'(m-1) - r'$ for some integers $0 < q' < q$ and $0 < r' \leq m$, which is Thing (II).

The case where S_e's wager at σ is less than $q - 1$ is similar. Suppose the wager is $(q-1) - k$ for some $k \geq 1$. Then by definition of X, $m' = m + 1$ and

$$s' = s + [(q-1) - k] = (qm - r) + q - k - 1 = q(m+1) - (r + k + 1).$$

If $r + k \leq m$, then $q' = q$ and $m' - r' = (m - r) - k$, satisfying Thing (I). If not, then $s' = q'(m+1) - r'$ for some integers $0 < q' < q$ and $0 < r' \leq m$, which gives Thing (II).

If index e does not receive attention at position n, then S_e's wager at σ is $q - 1$. In this case, $q' = q$ and $m' - r' = m - r$ regardless of whether the $(n+1)^{\text{st}}$ coin flip is heads or tails. Indeed if $X(n+1) = \mathtt{h}$, then

$$s' = s + (q-1) = (qm - r) + q - 1 = q(m+1) - (r+1),$$

and if $X(n+1) = \mathtt{t}$, then

$$s' = s - (q-1) = (qm - r) - q + 1 = q(m-1) - (r-1).$$

Based on the above calculations, we deduce that each index receives attention at most finitely many times over the course of the sequence X. Suppose this were not the case, and let e be the least index which receives attention infinitely often. Wait until a position n_0 such that no index lower than e ever again receives attention and $w(X \upharpoonright n_0) \geq e$. From then on, e receives attention at each position n where S_e wagers a value other than q because S_e is always the lowest-indexed strategy to do so. Things (I) and (II) then guarantee that the sequence $\{(q, m-r)\}_{n \geq n_0}$ lexicographically decreases each time e receives attention at position n while the sequence remains constant between other adjacent positions. Since e receives attention infinitely often and S_e is integer-valued, q eventually reaches the value 1 whereafter $m - r$ reaches 0. Once this happens, $\langle S_e, f_e \rangle$ is bankrupt. But a bankrupt gambling-and-savings strategy can no longer receive attention, a contradiction.

It is clear that once the last position has passed where some $j \leq e$ receives attention, M immediately achieves a new high water mark of $\$(e+1)$, if M did not already achieve this earlier, because the function w prevents any further gambling-and-savings strategy from receiving attention before this has happened. Thus M succeeds on X.

It remains to show that no gambling-and-savings strategy succeeds on X. The behavior of any gambling-and-savings strategy $\langle S_e, f_e \rangle$, once all gambling-and-savings strategies with index at most e have finished receiving attention, is simple: at position n, S_e always wagers $\$(q-1)$. Since such wagers preserve $q' = q$ and $m' - r' = m - r$, there are two possibilities. Either f_e only saves a finite amount of money, in which case $\langle S_e, f_e \rangle$ is certainly not successful, or else f_e saves an infinite amount. In the latter case, either Thing (I) or Thing (II) happens whenever f_e saves while q and r remain frozen at all other times. Therefore $\langle S_e, f_e \rangle$ eventually goes bankrupt. □

The anonymous referee asked whether we can obtain a similar paradox using more relaxed wager restrictions. In particular,

Question. *Does the Theorem above still hold if integer-valued gambling strategies are replaced with gambling strategies who wagers lie in the broader set of reals*

$$V = \{x : |x| \geq 1\} \cup \{0\}?$$

In general, determining whether there is additional power in gambling with wagers in V rather than integers remains an outstanding problem [1, 2].

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References

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