Prefix Distance Between Regular Languages

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For words $w_1, w_2 \in \Sigma^*$ and regular languages $L_1, L_2 \subseteq \Sigma^*$,

d	Levenshtein	prefix
$d(w_1, w_2)$	Wagner, Fisher (1974)	LCP
$d(w_1, L_2)$	Wagner (1974)	Bruschi, Pighizzini (2006)
$d(L_1, L_2)$	Mohri (2002)	$\times$
$d(L_1)$	Konstantinidis (2005)	$\times$

A distance is a function $d : \Sigma^* \times \Sigma^* \to [0, \infty)$ such that

1. $d(x, y) = 0$ if and only if $x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, y) \leq d(x, w) + d(w, y)$

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We can further generalize this to a distance between two languages L_1 and L_2 by

$$d(L_1, L_2) = \min\{d(w_1, w_2) \mid w_1 \in L_1, w_2 \in L_2\}.$$

The prefix distance of x and y counts the number of symbols which do not belong to the longest common prefix of x and y.

$$d_p(x, y) = |x| + |y| - 2 \cdot \max_{z \in \Sigma^*}\{|z| \mid x, y \in z\Sigma^*\}.$$

Harbord $\rightarrow$ Harbourfront

A nondeterministic finite automaton is a 5-tuples

$$A = (Q, \Sigma, \delta, I, F)$$

where

- Q is a finite set of states,
- Σ is the input alphabet,
- $\delta \subseteq Q \times \Sigma \times Q$ is the transition function,
- $I \subseteq Q$ is the set of initial states,
- $F \subseteq Q$ is the set of accepting states.

A weighted finite transducer is a 6-tuple

$$T = (Q, \Sigma, \Delta, I, F, E)$$

where

- Q is a finite set of states,
- Σ is the input alphabet,
- Δ is the output alphabet,
- $I \subseteq Q$ is the set of initial states,
- $F \subseteq Q$ is the set of accepting states,
- $E \subseteq Q \times (\Sigma \cup \{\varepsilon\}) \times (\Delta \cup \{\varepsilon\}) \times \mathbb{K} \times Q$ is the set of transitions with weights in the semiring $\mathbb{K}$.

The size of T, denoted $|T|$, is the sum of the number of states and transitions of T, $|Q| + |E|$.

A path or computation of T is a word π over the alphabet of transitions E

$$\pi = (p_1, u_1, v_1, w_1, q_1) \cdots (p_n, u_n, v_n, w_n, q_n)$$

with $q_i = p_{i+1}$ for $1 \leq i \leq n$.

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with $q_i = p_{i+1}$ for $1 \leq i \leq n$. Let $\omega : E^* \to \mathbb{K}$ be a weight function for paths $\pi = \pi_1 \cdots \pi_n$ defined by

$$\omega(\pi) = \sum_{i=1}^{n} \pi_i.$$

The label of a path π, denoted $\ell(\pi)$, is the pair of words (x, y) with $x = u_1 \cdots u_n$ and $y = v_1 \cdots v_n$.

The label of a path π, denoted $\ell(\pi)$, is the pair of words (x, y) with $x = u_1 \cdots u_n$ and $y = v_1 \cdots v_n$. We define a weight function $w : \Sigma^* \times \Sigma^* \to \mathbb{K}$ for labels (x, y) defined as the weight of the minimum weight accepted path labeled by (x, y),

$$w(x, y) = \min_{\pi \in E^*} \{\omega(\pi) \mid \ell(\pi) = (x, y)\}.$$

For an alphabet Σ, let $\mathcal{E}_\Sigma$ be the alphabet of edit operations over Σ,

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The cost of an edit string $e = e_1 e_2 \cdots e_n$ is the sum of the cost of each symbol

$$c(e) = \sum_{i=1}^{n} c(e_i).$$

We define the following sets of edit operations and their costs:

- identities $\mathcal{E}_0 = \{(a/a) \mid a \in \Sigma\}$ with cost 0.
- insertions $\mathcal{I} = \{(\varepsilon/a) \mid a \in \Sigma\}$ with cost 1,
- deletions $\mathcal{D} = \{(a/\varepsilon) \mid a \in \Sigma\}$ with cost 1,
- substitutions $\mathcal{S} = \{(a/b) \mid a \neq b, a, b \in \Sigma\}$ with cost 2,

We define the language of edit strings for the prefix distance L_p by

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We define the function $d'_p : \Sigma^* \times \Sigma^* \to \mathbb{N}$ on $x, y \in \Sigma^*$ by

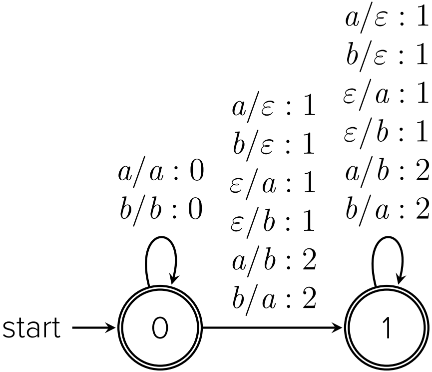
$$d'_p(x, y) = \min_{e \in L_p} \{ c(e) \mid h(e) = (x, y) \}.$$

Theorem

Let $x, y \in \Sigma^*$ be two words. Then $d'_p(x, y) = d_p(x, y)$.

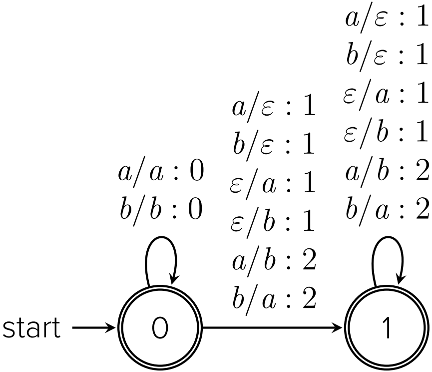
Consider an edit string $e \in L_p$ for two words $x = px'$ and $y = py'$ where p is the longest common prefix of x and y.

- Split e into two parts $e = e_0 e_1$ with $e_0 \in \mathcal{E}_0^*$ and $e_1 \in (\mathcal{E} \setminus \mathcal{E}_0)^*$.
- To minimize $c(e)$, e_0 must be as long as possible and e_1 as short as possible.
- Then e_1 corresponds to an edit string for x' and y'.
- Thus, $c(e) = c(e_1) = |x'| + |y'| = d_p(x, y)$.


a/ε : 1
b/ε : 1
ε/a : 1
ε/b : 1
a/b : 2
b/a : 2
a/a : 0
b/b : 0
a/ε : 1
b/ε : 1
ε/a : 1
ε/b : 1
a/b : 2
b/a : 2
start
0
1

Lemma

The set of accepting paths of the transducer T_p over Σ corresponds to exactly the set of edit strings over Σ belonging to L_p. If π is an accepting path of T_p and e_π is the corresponding edit string, then the weight of π is $c(e_\pi)$.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Lemma

Let $x, y \in \Sigma^*$. Then the weight $w(x, y)$ of x and y in T_p is exactly $d_p(x, y)$.

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- $w(x, y)$ is the weight of the minimal weight accepted path labeled by (x, y)
- The accepting paths of T_p corresponds to edit strings in L_p
- The minimal cost edit string in L_p for x and y has cost $d_p(x, y)$

The composition $T_1 \otimes T_2 = (Q, \Sigma, \Gamma, I, F, E)$ of two weighted transducers $T_1 = (Q_1, \Sigma, \Delta, I_1, F_1, E_1)$ and $T_2 = (Q_2, \Delta, \Gamma, I_2, F_2, E_2)$ is defined by

- $Q = Q_1 \times Q_2$,
- $I = I_1 \times I_2$,
- $F = Q \cap (F_1 \times F_2)$,
- and the transition set E consists of transitions of the form $((q_1, q_1'), a, c, w_1 + w_2, (q_2, q_2'))$ for each transition $(q_1, a, b, w_1, q_2) \in E_1$ and $(q_1', b, c, w_2, q_2') \in E_2$.

The composition $T_1 \otimes T_2$ can be computed in $O(|T_1||T_2|)$ time.

Theorem

Let L_1 and L_2 be regular languages recognized by NFAs A_1 and A_2, respectively. If $x \in L_1$ and $y \in L_2$, then (x, y) is the label of an accepting path of $T = A_1 \otimes T_p \otimes A_2$ and the weight of (x, y) in T is $d_p(x, y)$.

For any accepting path of T,

- the input part must be recognized by A_1,
- the output part must be recognized by A_2,
- the path must correspond to an edit string in L_p.

Thus, there is an accepting path labeled (x, y) with weight $d_p(x, y)$.

Theorem

For given NFAs A_1 and A_2 recognizing the languages L_1 and L_2, respectively, the value $d_p(L_1, L_2)$ can be computed in polynomial time.

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- Transducer composition can be done in time $O(|A_1||A_2|)$.
- Shortest path can be computed in time polynomial in the size of $A_1 \otimes T_p \otimes A_2$.

The inner distance of a language L, also called the self distance is the minimal distance between any two distinct words that belong to L,

$$d(L) = \min\{d(w_1, w_2) \mid w_1, w_2 \in L, w_1 \neq w_2\}.$$

To compute the inner distance, we can use the same approach, but we must exclude all edit strings with cost 0.

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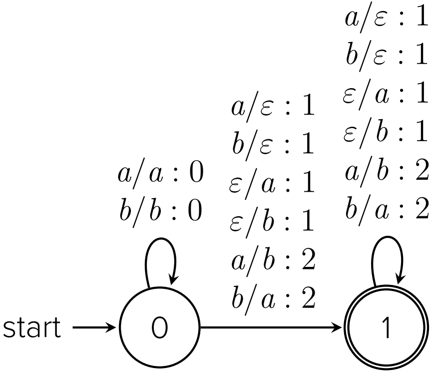

We can apply the same approach to suffix distance.

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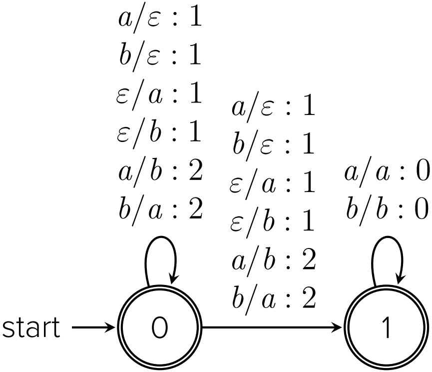

And infix distance.

$$L_f = (\mathcal{E} \setminus \mathcal{E}_0)^* \mathcal{E}_0^* (\mathcal{E} \setminus \mathcal{E}_0)^*$$

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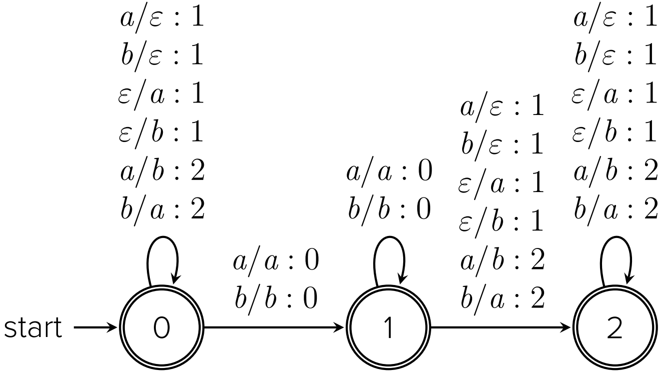

We have shown

- how to compute the prefix distance between two regular languages
- how to compute the inner prefix distance of a regular language
- how to compute the suffix and infix distances for the above

- What about other distances?
- What about the distance between a regular language and a context-free language?