Section 2.3, #8.

(a) The vectors in the intersection are members of both sets. Therefore, they can be expressed as either \[
\begin{pmatrix}
t - u \\
3t + u \\
u
\end{pmatrix}
\quad \text{or as} \quad
\begin{pmatrix}
4r - s \\
0 \\
s + r
\end{pmatrix}
\]. Setting these equal gives 3 equations, one per row: thus

\[
\begin{pmatrix}
t - u \\
3t + u \\
u
\end{pmatrix} =
\begin{pmatrix}
4r - s \\
0 \\
s + r
\end{pmatrix}
\]

becomes

\[
t - u = 4r - s \\
3t + u = 0 \\
u = s + r.
\]

Now, set this up in standard matrix form:

\[
\begin{pmatrix}
4 & -1 & -1 & 1 \\
0 & 0 & 3 & 1 \\
1 & 1 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
r \\
s \\
t
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}.
\]

Now, let us row-reduce this. I’ll let you figure out the justifications for the steps.

\[
\begin{pmatrix}
4 & -1 & -1 & 1 \\
0 & 0 & 3 & 1 \\
1 & 1 & 0 & -1
\end{pmatrix} \rightarrow
\begin{pmatrix}
1 & 1 & 0 & -1 \\
0 & 0 & 3 & 1 \\
4 & -1 & -1 & 1
\end{pmatrix} \rightarrow
\begin{pmatrix}
1 & 1 & 0 & -1 \\
0 & 0 & 3 & 1 \\
0 -5 & -1 & 1
\end{pmatrix} \rightarrow
\begin{pmatrix}
1 & 1 & 0 & -1 \\
0 & 1 & 1/5 & -1 \\
0 & 0 & 1 & 1/3
\end{pmatrix}.
\]

Now that the matrix is in row-echelon form, we can see that \( t = -1/3u \), and can also find expressions for \( r \) and \( s \) by back-substituting. Once we do this, and write both expressions in terms of \( u \), they each become:

\[
\begin{pmatrix}
-4u/3 \\
0 \\
u
\end{pmatrix}.
\]

This is a line through the origin \((0, 0, 0)\), with direction vector \((-4/3, 0, 1)\).
(b) Here the procedure is the same. Setting the components equal and converting to matrix form yields
\[
\begin{pmatrix}
4 & -1 & -1 & -1 \\
1 & 0 & -1 & 0 \\
1 & 1 & 0 & -1
\end{pmatrix}
\begin{pmatrix}
r \\
s \\
t \\
u
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}.
\]

It’s unusual that the right-hand side wound up being all zeros, but it really doesn’t matter. We still just row-reduce. This eventually gets us to:
\[
\begin{pmatrix}
1 & 0 & 0 & -1/2 \\
0 & 1 & 0 & -1/2 \\
0 & 0 & 1 & -1/2
\end{pmatrix}
\begin{pmatrix}
r \\
s \\
t \\
u
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}.
\]

(This time I used Gauss-Jordan elimination, getting all the way to reduced row-echelon form.) The solution set to this matrix can now be written as \( r = s = t = u/2 \), where \( u \) is a free variable. Plugging back into the original expressions, both forms become
\[
\begin{pmatrix}
3u/2 + 1 \\
u/2 \\
u + 1
\end{pmatrix}
\]

This is a line through the point \((1, 0, 1)\), with direction vector \((3/2, 1/2, 1)\).
Section 3.1, #4.

(a) We are given the system

\[3X_1 + X_2 - X_3 = 0\]
\[X_1 - 2X_2 - X_4 = 0\]
\[X_1 + 3X_2 = 0.\]

In augmented matrix form, this is

\[
\begin{pmatrix}
3 & 1 & -1 & 0 & 0 \\
1 & -2 & 0 & -1 & 0 \\
1 & 3 & 0 & 0 & 0
\end{pmatrix}
\]

(Note that we could have left off the constants column this time, since they are all zeros, and will stay that way.)

To solve this problem, we will row-reduce. This time, I will include the justifications (i.e. which elementary operation is being done) for each step.

\[
\begin{pmatrix}
3 & 1 & -1 & 0 & 0 \\
1 & -2 & 0 & -1 & 0 \\
1 & 3 & 0 & 0 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & -2 & 0 & -1 & 0 \\
0 & 3 & 1 & -1 & 0 \\
0 & 3 & 0 & 0 & 0
\end{pmatrix}
\]

\[
R_3 \rightarrow R_3 - 3R_1
\]

\[
R_2 \rightarrow R_2 + 3R_1
\]

\[
R_3 \rightarrow R_3 - 5R_2
\]

\[
R_1 \rightarrow R_1 - 2R_3/7
\]

\[
R_2 \rightarrow R_2 + 3R_3/7
\]

Thus we can take \(X_4 = t\) to be any real number, and the solution set is written as

\[
\left\{ \begin{pmatrix}
21t/35 \\
-t/5 \\
8t/5 \\
t
\end{pmatrix} \mid t \in \mathbb{R} \right\}
\]

Now, this solution is rather unsatisfactory, because of all the fractions. The next page shows a much nicer solution.

Continued on next page.
Let us back up several steps. After 2 steps of row-reduction, we had reduced to:

\[
\begin{pmatrix}
1 & -2 & 0 & -1 & 0 \\
0 & 7 & -1 & 3 & 0 \\
0 & 5 & 0 & 1 & 0
\end{pmatrix}
\]

This would almost be in row-echelon form if we changed the order of columns (i.e. the order of the variables. Let us do so.

\[
\begin{pmatrix}
1 & 0 & -1 & -2 & 0 \\
0 & -1 & 3 & 7 & 0 \\
0 & 0 & 1 & 5 & 0
\end{pmatrix}
\]

Now, it is crucial to remember that the columns now correspond, in order, to the variables \(X_1, X_3, X_4, X_2\). If we finish Gauss-Jordan elimination, we get:

\[
\begin{pmatrix}
1 & 0 & 0 & 3 & 0 \\
0 & 1 & 0 & 8 & 0 \\
0 & 0 & 1 & 5 & 0
\end{pmatrix}
\]

So, this means the 4th variable, \(X_2\), can be taken to be free, and, if we write \(X_2 = u\), the general solution is \((X_1, X_2, X_3, X_4) = (-3u, u, -8u, -5u)\), where \(u\) may be any real number. In set notation, this is written

\[
\left\{ \begin{pmatrix}
-3 \\
1 \\
-8 \\
-5
\end{pmatrix} \quad \left| \begin{array}{c}
u \\
u \in \mathbb{R}
\end{array} \right. \right\}.
\]

This may be thought of as a line through the origin in \(\mathbb{R}^4\) (although our geometric intuition tends to do poorly for dimensions greater than 3).

*This is the same solution* as that derived on the previous page. Setting \(t = -5u\) and simplifying shows that they are the same. In fact, this causes me to notice an earlier oversight: in the solution on the previous page, I failed to reduce a fraction: \(21t/35\) should have been written as \(3t/5\).

The solution on this page was much easier to calculate because of our better choice of column order.
(b) One thing you had to watch out for in this one, is that the typesetting was a bit misleading. Make sure you copied it down correctly! In augmented matrix form, it is:

\[
\begin{pmatrix}
2 & -1 & 0 & 0 & 0 \\
1 & 0 & -2 & 4 & 1 \\
0 & 1 & 2 & -3 & 2
\end{pmatrix}
\]

The quickest way is to begin by switching rows twice, yielding:

\[
\begin{pmatrix}
1 & 0 & -2 & 4 & 1 \\
0 & 1 & 2 & -3 & 2 \\
2 & -1 & 0 & 0 & 0
\end{pmatrix}
\]

After 2 more steps of Gaussian elimination, we achieve echelon form:

\[
\begin{pmatrix}
1 & 0 & -2 & 1 & 1 \\
0 & 1 & 2 & -3 & 2 \\
0 & 0 & 6 & -5 & 0
\end{pmatrix}
\]

Actually, I should divide the last row by 6 to get to echelon form, but there is no real reason to do this. Already, we can see that \(X_4\) can be taken as a free variable. To avoid fractions, let us call \(X_4 = 6t\). This allows us to find \(X_3 = 30t/6 = 5t\) without fractions. Continuing, \(X_2 = 2 - 2X_3 + 3X_4 = 2 + 8t\), and \(X_1 = 1 + 2X_3 - X_4 = 1 + 4t\). So the solution set is

\[
\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 4 \\ 8 \\ 5 \\ 6 \end{pmatrix} \right\} \quad t \in \mathbb{R}
\]